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ON THE ANALOGUES RELATING FLEXURE AND EXTENSION OF FLAT PLATES

By R. V. SOUTHWELL (*Oxford*)

[Received 10 January 1950]

SUMMARY

The displacement of a flat plate bent by transverse loading, and the extensional (Airy) stress-function in 'plane strain' or in 'plane stress', are governed by equations of identical form; and the boundary conditions have identical form when edge-displacements are specified in the flexural, edge-tractions in the extensional problem, so mathematically, in these circumstances, only a single problem is presented. This, the 'first analogue' relating flexure and extension, is well known.

A 'second analogue', relating the flexural problem when edge-tractions with the extensional problem when edge-displacements are specified, is believed to have been first propounded in 1941. By introducing two quantities U and V, analogous with the components u and v of extensional displacement, it permits a treatment of the flexural problem by any method—e.g. 'two-diagram (relaxational) technique'—which yields extensional solutions of this second type.

In this paper both analogues are combined in an inclusive statement covering the perforated (multiply connected) plates which were discussed in 1948. Reasons are stated for believing that 'two-diagram technique' is preferable in problems governed by 'mixed' boundary conditions.

Nomenclature

$2h$ denotes the (uniform) thickness of the plate.

ρ denotes the uniform density of its material.

E denotes the Young's Modulus of its material.

μ denotes the Modulus of Rigidity of its material.

σ denotes the Poisson's Ratio of its material.

ν denotes the direction of the outward normal to an edge.

s denotes distance measured along the edge.

(x, ν) denotes the angle (measured counter-clockwise) between the x-direction and the direction ν.

$$[l \equiv \cos(x, \nu), \qquad m \equiv \sin(x, \nu).]$$

Symbols special to the flexural problem:

D denotes the 'flexural rigidity' of plate $\left(=\dfrac{2Eh^3}{1-\sigma^2}\right)$,

w the transverse (flexural) displacement,

Z the surface-intensity of transverse loading,

ϕ a function such that $D\nabla^2\phi = Z$,

N_x, N_y the line-intensities of shearing action on sections perpendicular to Ox, Oy,

M_x, M_y the line-intensities of flexural couple on sections perpendicular to Ox, Oy,

H_x, H_y the line-intensities of torsional couple on sections perpendicular to Ox, Oy

$$[H_x = H_y = H, \text{ say (cf. section 6)}],$$

M_ν the line-intensity of flexural couple on an edge,

H_ν the line-intensity of torsional couple on an edge.

$$[lH+mM_y = lH_\nu+mM_\nu, \qquad lM_x+mH = lM_\nu-mH_\nu.]$$

U, V are 'quasi-displacements' defined in section 3.

Symbols special to the extensional problem:

A, k denote functions of σ depending on the state of stress,

u, v the component displacements of the middle surface in the directions Ox, Oy

$$\left[\Delta \equiv \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right],$$

Ω the potential of the conservative body-forces X, Y,

$\rho X = -\rho \dfrac{\partial \Omega}{\partial x}$ the component of body-force in the direction of x,

$\rho Y = -\rho \dfrac{\partial \Omega}{\partial y}$ the component of body-force in the direction of y,

χ, Airy's stress-function,

X_x, X_y, Y_y the components of stress in the plane of the middle-surface,

e_{xx}, e_{xy}, e_{yy} the components of strain in the plane of the middle-surface,

X_ν, Y_ν the components along Ox, Oy of tractions applied to the edge.

$$[X_\nu = lX_x+mX_y, \qquad Y_\nu = lX_y+mY_y.]$$

Introduction

1. By seeking to adapt the relaxational technique to computation of flexural and extensional stresses in flat plates, I came to consider (1938) the mathematical analogues which relate these physically independent problems. One was well known though seldom utilized, e.g. to solve flexural problems by photo-elastic methods; but a second (so far as I know) was first propounded in a paper written jointly with L. Fox (**1**). Here I attempt to present both in one comprehensive statement, and thereby to show that 'slices' and 'slabs'—that is, the extension and flexure of flat plates—are problems which should not be treated separately (as they commonly are) because mathematically they are identical.

2. The flexural displacement w is related with the line-intensities of bending couple (M_x, M_y) and of torsional couple $(H_x = H_y = H)$ by†

$$\left.\begin{aligned} M_x &= D\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right), \qquad H = D(1-\sigma)\frac{\partial^2 w}{\partial x \partial y}, \\ M_y &= D\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right), \end{aligned}\right\} \tag{1}$$

and these expressions substituted in the three equations of momental equilibrium yield expressions for the line-intensities of shearing action (N_x, N_y), together with the relation

$$D\nabla^4 w = Z, \tag{2}$$

in which Z stands for the surface-intensity of transverse loading. Z being specified, (2) governs w at points in the middle-surface of the plate; and when the boundary conditions relate to *displacement*, then w and its normal gradient $\partial w/\partial \nu$ have to take given values at the boundary.

The extensional stresses, expressed in terms of 'Airy's stress-function' χ by

$$X_x = \frac{\partial^2 \chi}{\partial y^2}+\rho\Omega, \qquad X_y = -\frac{\partial^2 \chi}{\partial x \partial y}, \qquad Y_y = \frac{\partial^2 \chi}{\partial x^2}+\rho\Omega, \tag{3}$$

satisfy the two stress-equations of equilibrium which hold in circumstances either of 'plane stress' $(Z_x = Z_y = Z_z = 0)$ or of 'plane strain'

$$(Z_x = Z_y = e_{zz} = 0);$$

and the conditions for compatibility of the accompanying strains require that χ shall satisfy the equation

$$\nabla^4 \chi + k\nabla^2(\rho\Omega) = 0, \tag{4}$$

where

$$\left.\begin{aligned} k &= (1-\sigma) \qquad \text{in circumstances of 'plane stress'} \\ k &= (1-2\sigma)/(1-\sigma) \text{ in circumstances of 'plane strain'} \end{aligned}\right\} \tag{5}$$

and when the body-forces are conservative and hence expressible in terms of a potential Ω (cf., e.g., (**3**), sections 401–6). When the *tractions* applied to the edge are specified, boundary values of $\partial\chi/\partial x$, $\partial\chi/\partial y$ can be computed from the expressions

$$-\frac{\partial \chi}{\partial x} = \int \{Y_\nu - \rho\Omega \sin(x,\nu)\}\, ds, \qquad \frac{\partial \chi}{\partial y} = \int \{X_\nu - \rho\Omega \cos(x,\nu)\}\, ds, \tag{6}$$

so χ and its normal gradient $\partial\chi/\partial\nu$ have to take given values on the boundary.‡ Accordingly (since the forms of (2) and (4) are exactly similar), *that case of the flexural problem in which edge-displacements, and that case of the extensional problem in which edge-tractions are specified, reduce to one*

† Cf. e.g. (**2**), section 298. This paper employs a slightly different notation.

‡ Cf. (**1**), 1941, section 7.

and the same mathematical problem. This is the first of the analogues mentioned in section 1.

3. When edge-displacements are specified in an extensional problem, use of the function χ is less convenient (at least in a relaxational treatment) because its *second* derivatives then enter into the boundary conditions; and a better procedure is to work in terms of component displacements u and v, related by the equations

$$\left.\begin{aligned} \frac{\partial \Delta}{\partial x} + A(\nabla^2 u + \rho X/\mu) &= 0, \\ \frac{\partial \Delta}{\partial y} + A(\nabla^2 v + \rho Y/\mu) &= 0, \end{aligned}\right\} \tag{7}$$

in which Δ stands for $\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)$ and

$$\left.\begin{aligned} A &= (1-\sigma)/(1+\sigma) \text{ in circumstances of 'plane stress'} \\ A &= (1-2\sigma) \qquad\quad \text{in circumstances of 'plane strain'.} \end{aligned}\right\} \tag{8}$$

The second analogue mentioned in section 1 relates this case of the extensional problem (case 'A': edge *displacements* specified) with that case ('B') of the flexural problem in which the line-intensities of force and couple (i.e. *tractions*) have to take given values at the boundary. It makes use of two quantities U and V which at all points in the middle surface of the plate have to satisfy†

$$\left.\begin{aligned} \frac{\partial \Delta'}{\partial x} + \frac{1+\sigma}{1-\sigma}\nabla^2 U + 2\frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial \Delta'}{\partial y} + \frac{1+\sigma}{1-\sigma}\nabla^2 V + 2\frac{\partial \phi}{\partial y} &= 0 \end{aligned}\right\} \tag{9}$$

when Δ' stands for $\left(\dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y}\right)$ and where ϕ *is a calculable function*. Their boundary values are to be calculated from

$$\left.\begin{aligned} -D\frac{\partial U}{\partial s} &= (M_\nu - D\phi)\sin(x,\nu) - \cos(x,\nu)\int\left(N_\nu - \frac{\partial H_\nu}{\partial s} + D\frac{\partial \phi}{\partial \nu}\right)ds, \\ D\frac{\partial V}{\partial s} &= (M_\nu - D\phi)\cos(x,\nu) + \sin(x,\nu)\int\left(N_\nu - \frac{\partial H_\nu}{\partial s} + D\frac{\partial \phi}{\partial \nu}\right)ds. \end{aligned}\right\} \tag{10}$$

† Cf. (**4**), equations (25)–(27)—which also carry numbers (with asterisks) referring to their derivation in (**1**). The quantities $\nabla^2 w_1$, Δ, G, H, N, in those equations, are here replaced by ϕ, Δ', $-M_\nu$, H_ν, N_ν respectively, and ϕ is *not* assumed to be zero on the boundary. (Cf. sections 7, 8 and 11.)

Clearly, (7) and (9) are exactly analogous except that the constant A, in (7), is replaced by a different constant in (9); and this change amounts (in effect) to substitution of a fictitious value σ' for Poisson's ratio σ. We shall find (section 10) that the substituted value implies mechanical stability of the presumed material; so nothing obstructs a solution of case 'B' of the flexural problem by any method that has been successful in case 'A' of the extensional problem.

4. In 1948 I extended this 'treatment by analogy' to plates which have internal boundaries (holes) either loaded or unloaded. I showed that these entail no serious complication of the relaxational technique, being tractable by a device ('block relaxation') which has long been customary; and exemplified the requisite computations by a solution, obtained by Miss E. D. Lang and Miss G. Vaisey, for a tension-member ('tie') to which load is applied through 'pins'. The 'isochromatic lines' of the photo-elastic method were deduced and presented in a diagram ((**4**), Fig. 7) which gave them in somewhat insufficient detail. More recently further 'lines' have been constructed by Miss Vaisey, whose more complete diagram is reproduced as one part of Plate I. For the accompanying photo-elastic photograph I am indebted to Dr. M. M. Frocht: the uniform longitudinal traction on a half-circumference, which was the mode of loading presumed in the computations (Plate I *a*), can hardly have been exactly reproduced; but 'Saint-Venant's principle' justifies a comparison, and the correspondence is seen to be very close.

5. Neither in 1941 nor 1948 was detailed consideration given to 'mixed' boundary conditions (specifying displacements at some, tractions at other points on the edge), because no practically important case had been confronted. Two distinct relaxational techniques were available, either of which could deal with 'mixed' conditions,† and the question of their relative merits was "left to the test of experience". But by 1948 it could be claimed that with their aid *any* problem concerned with flat elastic plates is soluble with sufficient accuracy for practical purposes; and as regards flexure this position would not have been attained without understanding of the 'second analogue' (section 3).

But the method employed to derive this second analogue was not made evident in the papers of 1941 and 1948; the significance of the function ϕ (section 3) is not apparent, and the derivation of (10) seems artificial. Moreover, the two analogues are related: *they can be combined in a single generalization*. This, the main thesis of the present paper, is established

† The application of one of these (viz. 'two-diagram technique') has been exemplified by L. Fox (**5**).

in sections 7–9. It is utilized in discussions of 'mixed' boundary conditions (sections 11–12) and of 'perforated' plates (sections 13–14).

My thanks are due to Miss G. Vaisey for her assistance, and to the Clothworkers' Company whose grant made that assistance possible. I am indebted to her both for the drawing in Plate I and for the computation and plotting which this entailed. (The earlier diagram mentioned in section 4 was drawn by Mr. D. N. de G. Allen.)

First extensional-flexural analogue ('Case A' of flexure and 'Case B' of extension)†

6. The first analogue used in the paper of 1941 is based on the similarity of equations (2) and (4). Of these the former is derived from the conditions of flexural equilibrium, viz.

$$\left.\begin{aligned} Z+\frac{\partial N_x}{\partial x}+\frac{\partial N_y}{\partial y}&=0\\ N_x+\frac{\partial M_x}{\partial x}+\frac{\partial H_y}{\partial y}&=0\\ N_y+\frac{\partial H_x}{\partial x}+\frac{\partial M_y}{\partial y}&=0 \end{aligned}\right\} \qquad (11)$$

(in which $H_x = H_y = H$, say), by substituting for M_x, M_y, H from (1). Thereby we obtain the expressions

$$-N_x = D\frac{\partial}{\partial x}(\nabla^2 w), \qquad -N_y = D\frac{\partial}{\partial y}(\nabla^2 w), \qquad (12)$$

and these substituted in the first of (11) yield (2).

The second (extensional) equation (4) is obtained from the condition of compatibility

$$\frac{\partial^2}{\partial y^2}e_{xx}-\frac{\partial^2}{\partial x\partial y}e_{xy}+\frac{\partial^2}{\partial x^2}e_{yy}=0 \qquad (13)$$

by substituting for e_{xx}, e_{xy}, e_{yy} from

$$e_{xx}=\frac{1}{E}[X_x-\sigma(Y_y+Z_z)], \text{ etc.}, \qquad e_{xy}=\frac{2(1+\sigma)}{E}X_y, \qquad (14)$$

and then for X_x, X_y, Y_y their expressions (3). Two values (5) are taken by the constant k in (4) because $Z_z = 0$ in circumstances of 'plane stress', $e_{zz} = 0$ in circumstances of 'plane strain'.

It can be shown (cf. (8) of section 3) that the difference between plane stress and plane strain amounts in effect to an alteration in σ. On that

† The different 'cases' were thus denoted in (**1**) (section 8), but were given different letters in (**4**).

understanding only one sort of extensional system need be considered, and *we shall concentrate on conditions of plane stress*. Here (Z_z being zero) substitution from (3) for X_x, X_y, Y_y transforms the relations (14) to

$$\left.\begin{aligned} Ee_{xx} &= \nabla^2\chi-(1+\sigma)\frac{\partial^2\chi}{\partial x^2}+(1-\sigma)\rho\Omega, \\ -Ee_{xy} &= 2(1+\sigma)\frac{\partial^2\chi}{\partial x\partial y}, \\ Ee_{yy} &= \nabla^2\chi-(1+\sigma)\frac{\partial^2\chi}{\partial y^2}+(1-\sigma)\rho\Omega, \end{aligned}\right\} \tag{15}$$

and (13), when these expressions are substituted, becomes

$$\nabla^2\{\nabla^2\chi+(1-\sigma)\rho\Omega\} = 0, \tag{16}$$

—agreeing with (4) and (5) in circumstances of plane stress.

The formal similarity of (2) and (16) is evident, so if the boundary conditions are also similar (as they are in the two cases now under consideration) the flexural and extensional problems are mathematically analogous. Exact correspondence will obtain when

$$\frac{Dw}{\chi} = \frac{-Z}{(1-\sigma)\nabla^2(\rho\Omega)} = \text{const.} \tag{17}$$

(The constant is not necessarily 'non-dimensional'.)

Second extensional-flexural analogue ('Case B' of flexure and 'Case A' of extension)†

7. We proceed to explain the derivation of the 'second analogue' (section 3). Let ϕ be a function defined by

$$D\nabla^2\phi = Z \tag{18}$$

and by a boundary condition to be stated later. Then, when N_x, N_y are eliminated from (11), there results

$$\frac{\partial^2}{\partial x^2}(M_x-D\phi)+2\frac{\partial^2 H}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}(M_y-D\phi) = 0, \tag{19}$$

and comparing this with (13) we observe that

$$\left.\begin{aligned} (M_y-D\phi) \text{ corresponds with } e_{xx} &= \frac{\partial u}{\partial x}, \\ (M_x-D\phi) \text{ corresponds with } e_{yy} &= \frac{\partial v}{\partial y}, \\ -2H \text{ corresponds with } e_{xy} &= \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}. \end{aligned}\right\} \tag{20}$$

† Cf. footnote to section 6.

The correspondences suggest that we express $(M_y - D\phi)$, $(M_x - D\phi)$, $-2H$ by $D\dfrac{\partial U}{\partial x}$, $D\dfrac{\partial V}{\partial y}$, $D\left(\dfrac{\partial V}{\partial x}+\dfrac{\partial U}{\partial y}\right)$ respectively, thereby introducing two quantities U and V which are the analogues of u and v. In this way, giving their expressions (1) to M_x, M_y, H, we obtain relations as under in U and V:

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}-\phi, \\ \frac{\partial V}{\partial y} &= \frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}-\phi, \\ \frac{\partial V}{\partial x}+\frac{\partial U}{\partial y} &= -2(1-\sigma)\frac{\partial^2 w}{\partial x\partial y}. \end{aligned}\right\} \tag{21}$$

8. The most general solution of the third of (21) is

$$\left.\begin{aligned} U &= -(1-\sigma)\frac{\partial w}{\partial x}+\frac{\partial \psi}{\partial x}, \\ V &= -(1-\sigma)\frac{\partial w}{\partial y}-\frac{\partial \psi}{\partial y}, \end{aligned}\right\} \tag{22}$$

ψ denoting some function of x and y as yet undefined; and when these expressions are substituted in the first and second of (21) we have

$$-\frac{\partial^2 \psi}{\partial y^2}=\frac{\partial^2 \psi}{\partial x^2}=\nabla^2 w-\phi, \tag{23}$$

whence $\nabla^2\psi = 0$ and so (9) result when w is eliminated between those two equations and (22). Also, from (22) again,

$$\begin{aligned} \frac{\partial}{\partial s}\left(\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\right) &= \left[\cos(x,\nu)\frac{\partial}{\partial y}-\sin(x,\nu)\frac{\partial}{\partial x}\right]\left(-2\frac{\partial^2\psi}{\partial x\partial y}\right) \\ &= 2\left[\cos(x,\nu)\frac{\partial}{\partial x}+\sin(x,\nu)\frac{\partial}{\partial y}\right](\nabla^2 w-\phi), \text{ by (23)}, \\ &\equiv 2\frac{\partial}{\partial \nu}(\nabla^2 w-\phi). \end{aligned} \tag{24}$$

Now $\quad N_\nu = \cos(x,\nu)N_x+\sin(x,\nu)N_y = -D\dfrac{\partial}{\partial\nu}\nabla^2 w$, by (12).

Hence and from (24) we have

$$D\left(\frac{\partial U}{\partial y}-\frac{\partial V}{\partial x}\right)=2\int\left(N_\nu+D\frac{\partial\phi}{\partial\nu}\right)ds,$$

and from this and the expressions in section 7,

$$D\frac{\partial U}{\partial y}=\int\left(N_\nu+D\frac{\partial\phi}{\partial\nu}\right)ds-H, \qquad -D\frac{\partial V}{\partial x}=\int\left(N_\nu+D\frac{\partial\phi}{\partial\nu}\right)ds+H,$$

with $$D\frac{\partial U}{\partial x} = M_y - D\phi, \qquad D\frac{\partial V}{\partial y} = M_x - D\phi.$$

So
$$\left.\begin{aligned} D\frac{\partial U}{\partial s} &\equiv D\left\{\cos(x,\nu)\frac{\partial U}{\partial y} - \sin(x,\nu)\frac{\partial U}{\partial x}\right\} \\ &= \cos(x,\nu)\left\{\int\left(N_\nu + D\frac{\partial \phi}{\partial \nu}\right)ds - H\right\} - \sin(x,\nu)(M_y - D\phi), \end{aligned}\right\}$$

and similarly

$$\left.\begin{aligned} D\frac{\partial V}{\partial s} = \cos(x,\nu)(M_x - D\phi) + \sin(x,\nu)\left\{\int\left(N_\nu + D\frac{\partial \phi}{\partial \nu}\right)ds + H\right\}. \end{aligned}\right\} \tag{25}$$

But equilibrium of an element at the boundary requires that

$$\left.\begin{aligned} \cos(x,\nu)H + \sin(x,\nu)M_y &= \cos(x,\nu)H_\nu + \sin(x,\nu)M_\nu, \\ \cos(x,\nu)M_x + \sin(x,\nu)H &= \cos(x,\nu)M_\nu - \sin(x,\nu)H_\nu. \end{aligned}\right\} \tag{26}$$

Consequently the relations (25) are equivalent to

$$\left.\begin{aligned} D\frac{\partial U}{\partial s} &= \cos(x,\nu)\int\left(N_\nu - \frac{\partial H_\nu}{\partial s} + D\frac{\partial \phi}{\partial \nu}\right)ds - \sin(x,\nu)(M_\nu - D\phi), \\ D\frac{\partial V}{\partial s} &= \cos(x,\nu)(M_\nu - D\phi) + \sin(x,\nu)\int\left(N_\nu - \frac{\partial H_\nu}{\partial s} + D\frac{\partial \phi}{\partial \nu}\right)ds, \end{aligned}\right\} \tag{27}$$

and these yield boundary values of U and V.†

As yet no boundary condition has been imposed on ϕ. If its values at the boundary are specified, its values elsewhere are determined by (18) and can be found (approximately) with the aid of Relaxation Methods; then both ϕ and $\partial\phi/\partial\nu$ will be known.

Combination of the two analogues

9. We have from (15) (in circumstances of 'plane stress'),

$$\left.\begin{aligned} \frac{\partial u}{\partial x} = e_{xx} &= \frac{1}{E}\left[\frac{\partial^2 \chi}{\partial y^2} - \sigma\frac{\partial^2 \chi}{\partial x^2} + (1-\sigma)\rho\Omega\right], \\ \frac{\partial v}{\partial y} = e_{yy} &= \frac{1}{E}\left[\frac{\partial^2 \chi}{\partial x^2} - \sigma\frac{\partial^2 \chi}{\partial y^2} + (1-\sigma)\rho\Omega\right], \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = e_{xy} &= -\frac{2(1+\sigma)}{E}\frac{\partial^2 \chi}{\partial x \partial y}, \end{aligned}\right\} \tag{28}$$

and comparing (28) with (21) we see that *U and V will correspond with u and v when*

$$\left.\begin{aligned} w &\text{ corresponds with } \chi/E, \\ -\phi &\text{ corresponds with } (1-\sigma)\rho\Omega/E, \\ -\sigma &\text{ corresponds with } \sigma. \end{aligned}\right\} \tag{29}$$

† Equations (10) and (27) are identical. We observe that N_ν and H_ν are associated in a term $\left(N_\nu - \frac{\partial H_\nu}{\partial s}\right)$, as in Kirchhoff's explanation of the double boundary condition (Cf. (2), section 297).

Then with the left-hand side of (16) there will correspond

$$E(\nabla^4 w - \nabla^2 \phi) = E(\nabla^4 w - Z/D), \quad \text{by (18)},$$

—that is, *equation* (2) *will correspond with* (16) *and so with* (4); also

$\Delta' \equiv \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y}$ will correspond with Δ in equations (7),

$\dfrac{1+\sigma}{1-\sigma}$ will correspond with $A \left(= \dfrac{1-\sigma}{1+\sigma} \right)$ in equation (7),

$2\left(\dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}\right)$ will correspond with $-2(1-\sigma)\dfrac{\rho}{E}\left(\dfrac{\partial \Omega}{\partial x}, \dfrac{\partial \Omega}{\partial y}\right) = A\dfrac{\rho}{\mu}(X, Y)$, (30)

—that is, *equations* (9) *will correspond with* (7). Thus except that the sign of σ is reversed, an exact correspondence has been established which embraces both of the preceding analogues.

10. It remains to show that the change in the sign of σ is acceptable, having regard to the fact (cf. (3), section 123) that a material would be mechanically unstable whose Poisson's ratio had a value outside of the range

$$-1 < \sigma < \tfrac{1}{2}; \tag{31}$$

for if Relaxation Methods are applied to an unstable system, special devices must be used in order to ensure that the solution will not diverge.

We have seen that equations (9) relate to a quasi-extensional system in which σ is replaced by a fictitious value $\sigma' = -\sigma$. Clearly, in order that

$$-1 < \sigma' < \tfrac{1}{2},$$

σ must have a value in the range

$$1 > \sigma > -\tfrac{1}{2}; \tag{32}$$

so in order that a material which itself is stable may yield a stable material for the quasi-extensional system, both of the criteria (31) and (32) must be satisfied.—that is to say, its σ must have a value in the narrower range

$$-\tfrac{1}{2} < \sigma < \tfrac{1}{2}. \tag{33}$$

Now in fact, while (31) is the range imposed by theory, (33) is satisfied by all materials which have practical interest. Accordingly it imposes no restriction on the use that can be made of our two analogues.

Mixed boundary conditions in the flexural problem

11. When tractions are specified at some, displacements at other parts of the edge (or boundary), we have a choice between treating flexural

systems in terms of w or of U and V, extensional systems in terms of χ or of u and v. Working flexural problems in terms of w we have to satisfy, at parts of the boundary where M_ν and $\left(N_\nu - \frac{\partial H_\nu}{\partial s}\right)$ are specified, conditions involving second and third derivatives of w; working them in terms of U and V we have to satisfy, at parts of the boundary where $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ are specified, conditions involving only first derivatives of U and V. On this account the second procedure would seem preferable in flexural problems where the conditions are 'mixed', and in extensional problems, similarly, working in terms of displacements u and v. We proceed to derive the requisite conditions at a boundary.

For flexure, writing (21), section 7, in the equivalent forms

$$\left.\begin{aligned}(1-\sigma^2)\frac{\partial^2 w}{\partial x^2} &= (1-\sigma)\phi + \frac{\partial V}{\partial y} - \sigma\frac{\partial U}{\partial x},\\ (1-\sigma^2)\frac{\partial^2 w}{\partial y^2} &= (1-\sigma)\phi + \frac{\partial U}{\partial x} - \sigma\frac{\partial V}{\partial y},\\ -2(1-\sigma)\frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y},\end{aligned}\right\} \tag{34}$$

and combining these with the identities

$$\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\right) \equiv l\frac{\partial^2 w}{\partial x \partial y} - m\frac{\partial^2 w}{\partial x^2}, \qquad \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\right) \equiv l\frac{\partial^2 w}{\partial y^2} - m\frac{\partial^2 w}{\partial x \partial y}, \tag{35}$$

in which l, m stand for $\cos(x,\nu)$, $\sin(x,\nu)$, we deduce that

$$\left.\begin{aligned}-2(1-\sigma^2)\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\right) &= \cos(x,\nu)(1+\sigma)\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) +\\ &\quad + 2\sin(x,\nu)\left\{\frac{\partial V}{\partial y} - \sigma\frac{\partial U}{\partial x} + (1-\sigma)\phi\right\},\\ 2(1-\sigma^2)\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\right) &= 2\cos(x,\nu)\left\{\frac{\partial U}{\partial x} - \sigma\frac{\partial V}{\partial y} + (1-\sigma)\phi\right\} +\\ &\quad + \sin(x,\nu)(1+\sigma)\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right).\end{aligned}\right\} \tag{36}$$

In these, the wanted boundary conditions in U and V, *we may give* ϕ *any boundary values which make for simplification.*

Mixed boundary conditions in the extensional problem

12. From (36), *merely by using the analogue*, we can deduce corresponding boundary conditions for extensional systems. (As before we assume

conditions of plane stress, for which the correspondence is as stated in (29), section 9.) Then we have expressions for $\frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial x}\right)$ and $\frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial y}\right)$ which when we replace $\frac{\partial \chi}{\partial x}, \frac{\partial \chi}{\partial y}$ by their expressions (6), become†

$$\left.\begin{aligned} 2\frac{1-\sigma^2}{E}X_r &= 2\cos(x,\nu)\left(\frac{\partial u}{\partial x}+\sigma\frac{\partial v}{\partial y}\right)+ \\ &\quad +(1-\sigma)\sin(x,\nu)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \\ 2\frac{1-\sigma^2}{E}Y_r &= 2\sin(x,\nu)\left(\frac{\partial v}{\partial y}+\sigma\frac{\partial u}{\partial x}\right)+ \\ &\quad +(1-\sigma)\cos(x,\nu)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \end{aligned}\right\} \tag{37}$$

—agreeing with equations (5) of a paper by L. Fox (**5**) which treats of 'mixed' conditions in extensional problems.

Conditions at internal boundaries (holes) in perforated plates

13. From (36), because $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ must be single-valued, it follows that the integrals

$$\oint\left\{\frac{\partial U}{\partial s}+\frac{\partial V}{\partial \nu}+\frac{m}{A}(\Delta'+2\phi)\right\}ds, \qquad \oint\left\{\frac{\partial U}{\partial \nu}-\frac{\partial V}{\partial s}+\frac{l}{A}(\Delta'+2\phi)\right\}ds \tag{38}$$

must vanish severally in respect of every interior boundary (hole); and then, in order that w may be single-valued, the integral

$$-\oint\left(l\frac{\partial w}{\partial y}-m\frac{\partial w}{\partial x}\right)ds=\oint\left(x\frac{\partial^2 w}{\partial s\partial x}+y\frac{\partial^2 w}{\partial s\partial y}\right)ds$$

—i.e. the integral

$$\oint\left\{x\left(\frac{\partial U}{\partial s}+\frac{\partial V}{\partial \nu}\right)+y\left(\frac{\partial V}{\partial s}-\frac{\partial U}{\partial \nu}\right)+\frac{mx-ly}{A}(\Delta'+2\phi)\right\}ds \tag{39}$$

must vanish in respect of every hole. (Here l, m stand for $\cos(x,\nu)$, $\sin(x,\nu)$, as before. A now denotes the ratio $(1+\sigma)/(1-\sigma)$.)

These are the conditions stated in equations (41) of Southwell (**4**). *They have no analogue in the extensional problem* because (ibid. section 8) $\chi, \frac{\partial \chi}{\partial x}, \frac{\partial \chi}{\partial y}$ need not be single-valued.

† The cancellation of the terms in Ω was to be expected, since the body-forces enter neither into the stress-strain relations nor into the equations for transformation of stress.

14. In extensional solutions the rotation and the displacements must be single-valued, and hence (ibid. sections 12–13) the conditions

$$\left.\begin{aligned} &\oint \frac{\partial}{\partial \nu}\{\nabla^2\chi+(1-\sigma)\rho\Omega\}\,ds = 0 \\ &\oint \left[y\frac{\partial}{\partial \nu}-\sin(x,\nu)\right]\{\nabla^2\chi+(1-\sigma)\rho\Omega\}\,ds = 0 \\ &\oint \left[x\frac{\partial}{\partial \nu}-\cos(x,\nu)\right]\{\nabla^2\chi+(1-\sigma)\rho\Omega\}\,ds = 0 \end{aligned}\right\} \tag{40}$$

must hold in respect of every internal boundary. Using the correspondence stated in (29), we obtain analogous flexural conditions as under:

$$\left.\begin{aligned} &\oint \frac{\partial}{\partial \nu}(\nabla^2 w-\phi)\,ds = 0, \\ &\oint \left[y\frac{\partial}{\partial \nu}-\sin(x,\nu)\right](\nabla^2 w-\phi)\,ds = 0, \\ &\oint \left[x\frac{\partial}{\partial \nu}-\cos(x,\nu)\right](\nabla^2 w-\phi)\,ds = 0. \end{aligned}\right\} \tag{41}$$

Relations equivalent to (41) are obtained (ϕ here replacing $\nabla^2 w_1$) when $\mathbf{Z}$, $\mathbf{L}$, $\mathbf{M}$ are eliminated from

$$\left.\begin{aligned} &\mathbf{Z} = -D\oint \frac{\partial}{\partial \nu}\nabla^2 w\,ds, \qquad \mathbf{L} = -D\oint \left[y\frac{\partial}{\partial \nu}-\sin(x,\nu)\right]\nabla^2 w\,ds, \\ &\mathbf{M} = D\oint \left[x\frac{\partial}{\partial \nu}-\cos(x,\nu)\right]\nabla^2 w\,ds, \end{aligned}\right\} \tag{42}$$

and

$$\left.\begin{aligned} &D\oint \frac{\partial}{\partial \nu}\nabla^2 w_1\,ds+\mathbf{Z} = 0, \\ &D\oint \left[y\frac{\partial}{\partial \nu}-\sin(x,\nu)\right]\nabla^2 w_1\,ds+\mathbf{L} = 0, \\ &D\oint \left[\cos(x,\nu)-x\frac{\partial}{\partial \nu}\right]\nabla^2 w_1\,ds+\mathbf{M} = 0, \end{aligned}\right\} \tag{43}$$

which are (6) and (32) of Southwell (**4**). They determine the constants A, B, C in the expression

$$\phi = A+Bx+Cy$$

which ϕ ($\equiv \nabla^2 w_1$) must have at every internal boundary. When they are satisfied, U and V will be acyclic (ibid. section 19).

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'Isochromatic' lines for a tension-member strained by loads applied through pins, as determined (*a*) by the photo-elastic method in the laboratory of Dr. M. M. Frocht (Illinois Institute of Technology) and (*b*) by relaxational computation (G. Vaisey).

PLATE I

THE BENDING OF A UNIFORMLY LOADED CLAMPED PLATE IN THE FORM OF A CIRCULAR SECTOR

By H. R. HASSÉ (*The University, Bristol*)

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SUMMARY

The deflexion of a uniformly loaded plate in the form of a semicircle clamped along its boundary is obtained by a method due to Weinstein. This problem requires the solution of the biharmonic equation $\Delta\Delta w = Z$, where Z is given, subject to the conditions that $w = 0$ and $\partial w/\partial N = 0$ on the boundary, N being the direction of the outward normal. The solution is expressed in the form

$$w = W + \sum_{m=1}^{\infty} A_m v_m,$$

where, writing ψ for W or v_m, ψ is found by solving (in succession) two harmonic equations of the forms $\Delta f = Z$, where Z may be zero, and $\Delta\psi = f$, where f and ψ have to satisfy certain boundary conditions. The constants A_m are then determined to satisfy the boundary condition $\partial w/\partial N = 0$.

Numerical calculations show that five or six terms of the series $\sum_{m=1}^{\infty} A_m v_m$ give a good approximation to the accurate value as judged by the closeness with which the approximate solution satisfies the boundary condition $\partial w/\partial N = 0$. The procedure to be adopted in the case of the general circular sector and for non-uniform loading is indicated briefly.

The connexion between the deflexion problem and that of plane strain in which the stress function ϕ satisfies the equation $\Delta\Delta\phi = 0$, where ϕ and $\partial\phi/\partial N$ have given values on the boundary, is discussed as a preliminary to the further consideration of the latter problem by a method of the same type.

THIS paper deals with the deflexion of a uniformly loaded plate in the form of a circular sector whose boundary is clamped, a problem which does not appear to have been investigated hitherto. The method used is that developed by Weinstein, originally for the purpose of calculating lower bounds for the modes of oscillation of clamped plates and applied later to the statical case of the deflexion of a uniform clamped square plate (**1**). The theoretical justification of the method is given in a further paper, in which the question of convergence and that of the connexion of the method with the equivalent problem in the calculus of variations are fully treated (**2**).

1. A brief résumé of the method is as follows. The differential equation for the deflexion w of a plate of area S is

$$\Delta\Delta w = q/D, \tag{1}$$

where Δ is Laplace's operator, q the load per unit area, and

$$D = 2Eh^3/3(1-\sigma^2)$$

in the usual notation, where $2h$ is the thickness of the plate. The conditions on the boundary C are

$$w = 0, \qquad \frac{\partial w}{\partial N} = 0, \tag{2}$$

where N is the outward normal to C.

The first step is to obtain the solution of (1) when the boundary condition $\partial w/\partial N = 0$ is replaced by the condition that $\Delta w = 0$ on C. This modified problem is equivalent to solving in succession two problems, each dealing with the equilibrium of a loaded membrane. Denoting the solution by W and putting $\Delta W = f$, the first of the above problems is that of finding f when $\Delta f = q'$ in S, $f = 0$ on C, where $q' \equiv q/D$, and the second is to find W when $\Delta W = f$ in S, $W = 0$ on C.

To solve the actual problem as specified in (1) and (2) we write as the mth approximation to the solution

$$w = W + \sum_{n=0}^{m} A_n v_n, \tag{3}$$

where A_n are constants, so that

$$\Delta w = \Delta W + \sum_{n=0}^{m} A_n p_n, \quad \text{where} \quad \Delta v_n = p_n$$

and, since

$$\Delta\Delta w = \Delta\Delta W + \sum_{n=0}^{m} A_n \Delta p_n, \quad \text{and} \quad \Delta\Delta w = \Delta\Delta W = q',$$

we take $\Delta p_n = 0$ so that p_n is a harmonic function in S. Again $\Delta\Delta v_n = 0$ and, since $w = 0$ and $W = 0$ on C, $v_n = 0$ on C.

It is now necessary to introduce the boundary condition $\partial w/\partial N = 0$ on C, which is done by writing it in the form

$$\int_C p_n \frac{\partial w}{\partial N}\, ds = 0 \tag{4}$$

for all n from 0 to m. This is equivalent to

$$\iint_S p_n \Delta w \, dS = 0, \tag{5}$$

by using Green's theorem in the form, for two dimensions,

$$\iint_S (\phi\Delta\psi - \psi\Delta\phi)\, dS = \int_C \left(\phi \frac{\partial\psi}{\partial N} - \psi \frac{\partial\phi}{\partial N}\right) ds,$$

putting $\phi = p_n$, $\psi = w$, and noting that $\Delta p_n = 0$ in S and $w = 0$ on C.

The condition (4) is a weaker condition than $\partial w/\partial N = 0$ on C, so that in the equivalent problem of the calculus of variations, viz. that

$$J \equiv \iint_S [(\Delta w)^2 - 2q'w]\, dS,$$

is a minimum for the exact solution obtained from (3) by making m tend to infinity, the minimum values of J obtained by taking $m = 0, 1, 2, 3, \ldots$ in succession will steadily increase. Thus in these successive approximations the minimum value of J for the exact solution is approached from below. If the problem were treated by the Rayleigh–Ritz method, which relaxes the constraints as the approximation proceeds, the minimum value of J corresponding to the exact solution would be approached from above.

Finally the values of the constants A_n are obtained from (3) and (5) so that

$$\iint_S p_n \Big(\Delta W + \sum_{s=0}^{m} A_s p_s\Big)\, dS = 0,$$

which, using the usual notation $\iint_S PQ\, dS \equiv (P, Q)$, is written in the form

$$(p_n, \Delta W) + \sum_{s=0}^{m} A_s (p_s, p_n) = 0 \qquad (6)$$

for all values of n from 0 to m.

Using Green's theorem again, putting $\phi = v_n$, $\psi = \Delta W$, it follows that

$$\iint_S p_n \Delta W\, dS = \iint_S v_n q'\, dS,$$

so that (6) becomes

$$(v_n, q') + \sum_{s=0}^{m} A_s (p_s, p_n) = 0 \qquad (7)$$

for all n from 0 to m. This gives m equations for the A_m ($m = 0, 1, 2, \ldots$) in the successive approximations to the exact solution.

2. The case of a semicircular sector of unit radius will now be considered in detail. If the angle θ is measured from the axis of symmetry and r is measured from the centre of the straight part of the boundary, the expression for W is, taking $q' = 1$,

$$W = \tfrac{1}{64} r^4 + \tfrac{1}{48} r^4 \cos 2\theta + \tfrac{1}{192} r^4 \cos 4\theta + \sum^{\infty} F_k r^{k+2} \cos k\theta - \sum^{\infty} G_k r^k \cos k\theta, \quad (8)$$

where $k = 1, 3, 5, \ldots$, and

$$F_k = \frac{\sin(\frac{1}{2}k\pi)}{\pi} \frac{1}{k(k+1)(k^2-4)}, \qquad G_k = \frac{\sin(\frac{1}{2}k\pi)}{\pi} \frac{k+6}{k(k+1)(k+2)(k^2-16)}.$$

This result is a special case of that given, for a *supported* semicircular plate, by Timoshenko (*Theory of Plates and Shells*, p. 272) except that the first three terms in (8) are there expressed as a Fourier series of the same type as in the remaining part of (8). The functions p_n are now to be chosen; they must form a complete set of harmonic functions in the polar coordinates

(r, θ) and the functions v_n are then to be found as the solutions of the equations $\Delta v_n = p_n$, $v_n = 0$ on C. The values of v_n were originally obtained by indirect and complicated methods and the corresponding p_n were then deduced, but the former were found to be linear functions of comparatively simple expressions which were therefore taken as the appropriate v_n, and then the p_n are the elementary harmonic functions $p_n = 4(n+1)r^n \cos n\theta$, where n is zero or a positive integer.†

When n is odd, the corresponding v_k are given by

$$v_k = (r^k - r^{k+2})\cos k\theta, \tag{9}$$

which vanish when $r = 1$ and when $\theta = \pm\frac{1}{2}\pi$, i.e. on the boundary C of the plate.

When n is even, the v_n are more complicated; for $n = 0, 2, 4, 6$ they are

$$\left.\begin{aligned} v_0 &= r^2(1+\cos 2\theta) - \textstyle\sum B_k r^k \cos k\theta \\ v_2 &= r^4(\cos 2\theta + \cos 4\theta) - \textstyle\sum C_k r^k \cos k\theta \\ v_4 &= r^6(\cos 4\theta + \cos 6\theta) - \textstyle\sum D_k r^k \cos k\theta \\ v_6 &= r^8(\cos 6\theta + \cos 8\theta) - \textstyle\sum E_k r^k \cos k\theta \end{aligned}\right\}, \tag{10}$$

where the infinite series are those required to make v_n zero when $r = 1$, their values being clearly zero for $\theta = \pm\frac{1}{2}\pi$. The constants in these series are

$$\left.\begin{aligned} B_k &= -\frac{4}{\pi}\sin(\tfrac{1}{2}k\pi)\,\frac{2^2}{k(k^2-2^2)} \\ C_k &= \frac{4}{\pi}\sin(\tfrac{1}{2}k\pi)\,\frac{(4^2-2^2)k}{(k^2-2^2)(k^2-4^2)} \\ D_k &= -\frac{4}{\pi}\sin(\tfrac{1}{2}k\pi)\,\frac{(6^2-4^2)k}{(k^2-4^2)(k^2-6^2)} \\ E_k &= \frac{4}{\pi}\sin(\tfrac{1}{2}k\pi)\,\frac{(8^2-6^2)k}{(k^2-6^2)(k^2-8^2)} \end{aligned}\right\}. \tag{11}$$

It is possible that linear combinations of the p_n could have been chosen as new functions (P_n, say) and that, with these P_n and the corresponding V_n, the approximation (3) to the exact solution might give a more accurate result. This could only be tested by numerical calculation and it will be seen that the convergence is fairly satisfactory with the p_n as given above.

It will be noticed that the approximate solution (3) is composed of terms of the form $r^n \cos n\theta$ and $r^{n+2}\cos n\theta$ which are elementary solutions of the equation $\Delta\Delta\phi = 0$. The effect of Weinstein's method is to enable these terms, or combinations of them, to be arranged in an order of importance, though this cannot be done without introducing infinite series as these occur in the v_n when n is even, and also in W.

† In the application to the semicircular sector k will always denote an odd positive integer. In all cases m, n are positive integers, including zero.

3. Integrating over half the area, because of the symmetry, the value of (p_m, p_m) is $16(m+1)\pi/8$, except when $m = 0$, in which case

$$(p_0, p_0) = 16\pi/4.$$

If m, n are both even or both odd and $m \neq n$, $(p_m, p_n) = 0$. If m is even and n is odd,

$$(p_m, p_n) = \frac{16(m+1)(n+1)}{m+n+2} \cdot \frac{n}{n^2-m^2} \cos(\tfrac{1}{2}m\pi)\sin(\tfrac{1}{2}n\pi).$$

When n is odd $$(1, v_n) = \frac{2\sin(\frac{1}{2}n\pi)}{n(n+2)(n+4)},$$

and when n is even

$$(1, v_0) = -0{\cdot}14877\ 84 \qquad (1, v_2) = -0{\cdot}03902\ 14$$
$$(1, v_4) = 0{\cdot}01414\ 71 \qquad (1, v_6) = -0{\cdot}00687\ 145$$

Owing to the factor 16 in (p_m, p_n) it is convenient to put $A'_n = 16A_n$ and then the values of $A'_0, \ldots, A'_6$, as the solutions of (7) for the successive approximations given by $m = 0, 1, 2, \ldots, 6$, are

$m=0$	$A'_0 = \cdot18943\ 05$		
$m=1$	$A'_0 = \cdot16218\ 40$	$A'_1 = -\cdot03209\ 92$	
$m=2$	$A'_0 = \cdot25233\ 20$	$A'_1 = \cdot07410\ 40$	$A'_2 = \cdot05828\ 29$
$m=3$	$A'_0 = \cdot27874\ 32$ $A'_3 = \cdot02913\ 34$	$A'_1 = \cdot11687\ 24$	$A'_2 = \cdot09823\ 98$
$m=4$	$A'_0 = \cdot28884\ 42$ $A'_3 = \cdot05267\ 79$	$A'_1 = \cdot13819\ 05$ $A'_4 = \cdot01164\ 33$	$A'_2 = \cdot12603\ 38$
$m=5$	$A'_0 = \cdot29323\ 97$ $A'_3 = \cdot06798\ 45$	$A'_1 = \cdot14845\ 24$ $A'_4 = \cdot02168\ 68$	$A'_2 = \cdot14124\ 96$ $A'_5 = \cdot00403\ 94$
$m=6$	$A'_0 = \cdot29416\ 171$ $A'_3 = \cdot07344\ 349$ $A'_6 = \cdot00117\ 822$	$A'_1 = \cdot15093\ 441$ $A'_4 = \cdot02657\ 847$	$A'_2 = \cdot14562\ 011$ $A'_5 = \cdot00710\ 309$

The successive approximations to the solution w are then

$$W, \qquad w_0 \equiv W + A_0 v_0, \qquad w_1 \equiv W + A_0 v_0 + A_1 v_1, \qquad \ldots,$$
$$w_6 = W + A_0 v_0 + A_1 v_1 + \ldots + A_6 v_6,$$

and should make the values of $\partial w/\partial N$ on the boundary approach zero as the number of the constants A_m increases.

On the boundary $r = 1$ the values of $\partial w/\partial N$ for $\theta = 0°, 15°, \ldots, 75°$ are given in the following table.

TABLE I

	$\theta = 0°$	$\theta = 15°$	$\theta = 30°$	$\theta = 45°$	$\theta = 60°$	$\theta = 75°$
W	$-\cdot019\ 960$	$-\cdot019\ 523$	$-\cdot018\ 174$	$-\cdot015\ 799$	$-\cdot012\ 189$	$-\cdot007\ 024$
$m=0$	$-\cdot002\ 751$	$-\cdot002\ 463$	$-\cdot001\ 594$	$+\cdot000\ 052$	$+\cdot001\ 768$	$+\cdot003\ 628$
$m=1$	$-\cdot001\ 214$	$-\cdot001\ 041$	$-\cdot000\ 504$	$+\cdot000\ 434$	$+\cdot001\ 767$	$+\cdot003\ 134$
$m=2$	$+\cdot001\ 197$	$+\cdot000\ 795$	$-\cdot000\ 157$	$-\cdot001\ 058$	$-\cdot001\ 083$	$+\cdot000\ 166$
$m=3$	$-\cdot000\ 251$	$-\cdot000\ 081$	$+\cdot000\ 231$	$+\cdot000\ 226$	$-\cdot000\ 125$	$-\cdot000\ 312$
$m=4$	$+\cdot000\ 078$	$+\cdot000\ 012$	$-\cdot000\ 059$	$-\cdot000\ 016$	$+\cdot000\ 073$	$-\cdot000\ 062$
$m=5$	$-\cdot000\ 022$	$+\cdot000\ 005$	$+\cdot000\ 037$	$-\cdot000\ 013$	$-\cdot000\ 012$	$+\cdot000\ 034$
$m=6$	$+\cdot000\ 002$	$-\cdot000\ 002$	$+\cdot000\ 016$	$+\cdot000\ 005$	$-\cdot000\ 008$	$+\cdot000\ 010$

Similarly the values of $\partial w/\partial n$ on the boundary $\theta = \frac{1}{2}\pi$, calculated for $r = 0, 0{\cdot}125, \ldots, 0{\cdot}875$, are shown in the next table.

TABLE II

$r =$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
$\frac{\partial W}{\partial N}$	−·024 757	−·024 250	−·022 747	−·020 308	−·017 039	−·013 095	−·008 699	−·004 168
$\frac{\partial w_0}{\partial N}$	−·004 658	−·004 340	−·003 413	−·001 966	−·000 159	+·001 750	+·003 342	+·003 835
$\frac{\partial w_1}{\partial N}$	−·005 543	−·005 229	−·004 313	−·002 880	−·001 082	+·000 838	+·002 488	+·003 141
$\frac{\partial w_2}{\partial N}$	−·001 379	−·001 273	−·000 963	−·000 480	+·000 124	+·000 765	+·001 298	+·001 449
$\frac{\partial w_3}{\partial N}$	−·000 402	−·000 348	−·000 199	+·000 012	+·000 230	+·000 385	+·000 399	+·000 213
$\frac{\partial w_4}{\partial N}$	−·000 108	−·000 094	−·000 054	+·000 008	+·000 079	+·000 134	+·000 128	+·000 066
$\frac{\partial w_5}{\partial N}$	+·000 010	+·000 004	−·000 009	−·000 018	−·000 013	+·000 012	+·000 040	+·000 016
$\frac{\partial w_6}{\partial N}$	+·000 031	+·000 021	−·000 004	−·000 027	−·000 032	−·000 007	+·000 033	+·000 031

These tables show that, taking $m = 5$ or 6, the approximation given by (3) is sufficient to give results to an accuracy, in general, of about 1 in 1,000, if the accuracy is measured by the ratio of $\partial w_6/\partial N$ to $\partial W/\partial N$. In the tables this ratio is somewhat smaller for $\theta = 75°$, $r = 0{\cdot}75$ and $r = 0{\cdot}875$, indicating that the approximation becomes worse as θ approaches 90° and r approaches 1. This is to be expected as it is known that at a sharp bend on the boundary w cannot be represented by an analytic function.

The maximum deflexion, which must lie on the radius $\theta = 0$, was obtained by calculating w for $r = 0{\cdot}1, 0{\cdot}2, \ldots, 0{\cdot}9$ and deriving from them the maximum value. For $m = 5$ this is 0·002 022 at $r = 0{\cdot}48594$, and for $m = 6$ it is 0·002 023 at $r = 0{\cdot}48587$.

4. For the general sector of angle 2α and radius a the expression for W corresponding to (8) is

$$\frac{1}{64}r^4 - \frac{1}{48}\frac{r^4\cos 2\theta}{\cos 2\alpha} + \frac{1}{192}\frac{r^4\cos 4\theta}{\cos 4\alpha} + \sum F_k\left(\frac{r}{a}\right)^{k+2}\cos k\theta - \sum G_k\left(\frac{r}{a}\right)^{k}\cos k\theta, \tag{12}$$

where $k = (2n+1)\pi/2\alpha$, $n = 0, 1, 2, \ldots$, and

$$F_k = \frac{a^4\sin k\alpha}{2\alpha k(k+1)(k^2-4)} = \frac{a^4\sin k\alpha}{2\alpha k}\,\frac{1}{12}\left(\frac{1}{k-2} - \frac{4}{k+1} + \frac{3}{k+2}\right),$$

$$G_k = \frac{a^4\sin k\alpha}{2\alpha}\,\frac{(k+6)}{k(k+1)(k+2)(k^2-16)}.$$

Critical cases arise when $\cos 2\alpha = 0$ or $\cos 4\alpha = 0$. Taking for example $\alpha = \frac{1}{4}\pi$, then $k = 2,\ 6,\ 10,\ldots$, so that $F_2 \to \infty$ as $k \to 2$. If $k = 2+\epsilon$, ϵ small, then since $2k\alpha = \pi$ as $n = 0$ for F_2, it will be found, considering the second term in (11) and the first term in $\sum F_k (r/a)^{k+2} \cos k\theta$ and making $\epsilon \to 0$, that these terms together lead to the expression

$$\frac{-13r^4 \cos 2\theta}{144\pi} + \frac{[r^4 \log(r/a)\cos 2\theta - r^4 \theta \sin 2\theta]}{12\pi}.$$

The remaining values of F_k and all the values of G_k are given by the above expressions for these coefficients. The values of v_m are determined in the same way as before from those of $p_m = 4(m+1)r^m \cos m\theta$. The result is

$$v_m = r^{m+2}\left[\cos m\theta - \frac{\cos m\alpha}{\cos(m+2)\alpha}\cos(m+2)\theta\right] + \sum H_k \left(\frac{r}{a}\right)^k \cos k\theta, \tag{13}$$

where the H_k are to be determined so that $v_m = 0$ when $r = a$. A critical case will arise when $\cos(m+2)\alpha = 0$ and has to be treated in the same way as indicated above. For certain values of m only one additional term in H_k is needed; for example, when $\alpha = \frac{1}{4}\pi$, the value of v_m for $m = 2,\ 6,\ 10,\ldots$, is $(r^{m+2} - a^2 r^m)\cos m\theta$, which is zero when $r = a$ and when $\theta = \frac{1}{4}\pi$, i.e. on the boundary C.

If the loading is not uniform but can be expressed in the form

$$q' = \sum r^m \cos n\theta$$

(to take the case of symmetrical loading) the determination of W and the functions v_s can be dealt with on the lines exemplified above. Critical cases will arise depending on the values of m and n and have to be treated specially, either directly in any particular case or by a limiting process from a general result.

5. Returning to the case of the uniformly loaded semicircular plate and putting $w = (r^4/64) + \phi$, then $\Delta\Delta\phi = 0$ and $\phi = -r^4/64$ on C. Again, since $\partial w/\partial N = 0$ on C, it follows that $\partial\phi/\partial r = -\frac{1}{16}r^3$ on $\theta = \pm\frac{1}{2}\pi$ and

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0 \quad \text{on} \quad r = 1.$$

Hence ϕ is a function satisfying $\Delta\Delta\phi = 0$ in S, $\phi = -r^4/64$ on C,

$$\frac{\partial\phi}{\partial N} = -\tfrac{1}{16}r^3 \text{ on } \theta = \pm\tfrac{1}{2}\pi, \quad \text{and} \quad \frac{\partial\phi}{\partial N} = 0 \text{ on } r = 1.$$

This function can therefore be identified with the stress function in plane stress or plane strain, in which on the boundary

$$\widehat{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = -\tfrac{1}{16}r^2, \qquad \widehat{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = 0$$

and

$$\widehat{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -\frac{3r^2}{16},$$

so that the plate is acted on by the uniform normal thrust $\frac{1}{16}$ on the semi-circle $r = 1$, and by the equilibrating normal thrust $\frac{3}{16}r^2$ on the straight edge of the plate.

It is evident that the solution of the problem of the deflexion of a clamped plate loaded in any *given* manner can be used to derive the stress function for a plate acted on by a derived system of stress on the boundary. The converse problem of finding the stress function ϕ for a *given* stress system on the boundary, i.e. for *given* values of ϕ and $\partial\phi/\partial N$ on C, requires further consideration; it does not appear to be in all cases directly and easily transformable into the problem of the deflexion of a clamped plate, loaded in a manner which is to be deduced from a knowledge of the given values of ϕ and $\partial\phi/\partial N$ on the boundary.

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ISOLATED FORCE PROBLEMS IN TWO-DIMENSIONAL ELASTICITY (I)†

By M. ROTHMAN (*Royal Technical College, Glasgow*)

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SUMMARY

Stevenson (**1**) has shown that the stresses and displacements in two-dimensional elasticity can be expressed in terms of two complex potentials $\Omega(z)$ and $\omega(z)$. In this paper Stevenson's method is used to solve the problems of one or more isolated forces on the boundaries of holes of various shapes in an infinite plate, agreement being reached with special cases previously solved by different methods.

1. Introduction

If D is the complex displacement $u+iv$, Θ and Φ the generalized plane-stress combinations $\breve{x}\breve{x}+i\breve{y}\breve{y}$ and $\breve{x}\breve{x}-\breve{y}\breve{y}+2i\breve{x}\breve{y}$, then in the absence of body forces we may write

$$8\mu D = \kappa\Omega(z)-z\overline{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}), \tag{1.1}$$

$$2\Theta = \Omega'(z)+\overline{\Omega}'(\bar{z}), \tag{1.2}$$

and

$$-2\Phi = z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z}), \tag{1.3}$$

where μ is the rigidity, one of the Lamé constants (λ, μ), and $\kappa = 3-4\sigma$, σ being the modified Poisson's ratio. Dashes denote derivatives.

2. The cyclic functions of the complex potentials

For single-valued displacements, i.e. solutions involving no dislocations, equation (1.1) gives

$$Cy\{\kappa\Omega(z)-z\overline{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z})\} = 0, \tag{2.1}$$

where $Cy\phi$ denotes the change in ϕ, a cyclic function, on passing once round a closed circuit so as to keep the area enclosed on the left.

Differentiating (2.1) and using the fact that the stresses being single-valued, Θ must also be single-valued, and therefore that

$$Cy\{\Omega'(z)+\overline{\Omega}'(\bar{z})\} = 0, \tag{2.2}$$

equation (2.1) gives

$$Cy\Omega'(z) = 0 \tag{2.3}$$

and therefore

$$Cy\bar{\omega}'(\bar{z}) = \kappa Cy\Omega(z). \tag{2.4}$$

If now $X+iY$ and M are the complex force and couple resultants at the origin equivalent to the stresses round a closed circuit, then (**1**)

$$X+iY = -\tfrac{1}{4}iCy\{\Omega(z)+z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})\}, \tag{2.5}$$

and

$$M+iM^* = \tfrac{1}{4}Cy\{\omega(z)-z\omega'(z)\}, \tag{2.6}$$

† Forms part of a thesis approved for the degree of Ph.D. by the University of London.

so that, if $F = X+iY$, then from (2.4) and (2.5)

$$F = -\tfrac{1}{4}i(\kappa+1)Cy\Omega(z) \quad \text{or} \quad Cy\Omega(z) = 4iF/(\kappa+1), \tag{2.7}$$

or changing the sign of i and using (2.4), $Cy\omega'(z) = -4i\kappa\overline{F}/(\kappa+1)$ whence from (2.6)

$$Cy\omega(z) = -4i\kappa\overline{F}z/(\kappa+1)+4(M+iM^*). \tag{2.8}$$

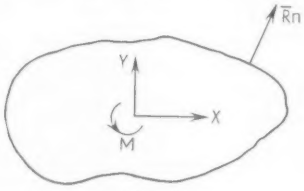

FIG. 1.

3. Form of the complex potentials for the infinite plate

We assume potentials of the form

$$\Omega(z) = A_0\log z+Az+\sum_1^\infty A_n z^{-n} \tag{3.1}$$

and

$$\omega(z) = Cz\log z+B_0\log z+Bz^2+\sum_1^\infty B_n z^{-n}, \tag{3.2}$$

where A is to be real since $\Omega = \alpha+i\beta z$, $\omega = \gamma+\delta z$ (β real) give rigid body displacements.

The terms in A and B give finite stresses at infinity (but will inevitably give rise to infinite displacements there) and there can be no other positive powers of z, the negative powers having less effect as we recede farther from the origin.

Assuming the force and couple resultants for the circuit at infinity to be F and M, equations (2.7) and (2.8) give $2\pi iA_0 = 4iF/(\kappa+1)$ and therefore $A_0 = 2F/\pi(\kappa+1)$ and $2\pi iCz+2\pi iB_0 = -4i\kappa\overline{F}z(\kappa+1)^{-1}+4(M+iM^*)$ and therefore $C = -2\kappa\overline{F}/\pi(\kappa+1)$, $B_0 = -2i(M+iM^*)/\pi$, so that

$$\Omega(z) = \frac{2F\log z}{\pi(\kappa+1)}+Az+\sum_1^\infty A_r z^{-r}, \tag{3.3}$$

and

$$\omega(z) = -\frac{2\kappa\overline{F}z\log z}{\pi(\kappa+1)}-(2iG/\pi)\log z+Bz^2+\sum_1^\infty B_r z^{-r}, \tag{3.4}$$

where M is the real part of G.

In particular for an unlimited plate with a force and couple nucleus F and G at the origin, we have

$$\Omega(z) = \frac{2F\log z}{\pi(\kappa+1)}, \tag{3.5}$$

and

$$\omega(z) = -\frac{2\kappa\overline{F}z\log z}{\pi(\kappa+1)}-\frac{2iG}{\pi}\log z. \tag{3.6}$$

4. Isolated forces on any boundary

Taking complex potentials

$$\Omega(z) = A\log z, \qquad \omega(z) = -\bar{A}z\log z, \tag{4.1}$$

then $$\breve{rr}+\breve{\theta\theta} = \Theta' = \Theta = \tfrac{1}{2}(A/z+\bar{A}/\bar{z}),$$

and $$\breve{rr}-\breve{\theta\theta}+2i\breve{r\theta} = \Phi' = \Phi e^{-2i\theta} = (\bar{z}/z)\Phi = \tfrac{1}{2}(\bar{A}/\bar{z}+A/z),$$

which is real, so that $\breve{r\theta} = 0$, and $\breve{rr}-\breve{\theta\theta} = \breve{rr}+\breve{\theta\theta}$, whence $\breve{\theta\theta} = 0$, and $\breve{rr} = \tfrac{1}{2}(A/z+\bar{A}/\bar{z})$. The potentials given by (4.1) are therefore seen to give a simple radial state of stress.

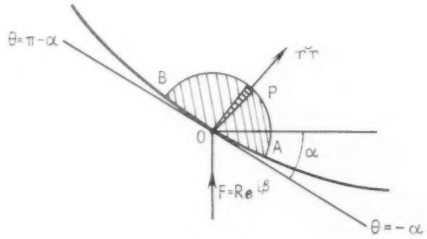

FIG. 2.

It is clear from the previous results that if the origin be taken inside the material, the solution will not give single-valued displacements and so will be non-physical. If, however, the origin be taken outside the material, then no circuit within the material being reducible to the origin, the objection does not arise, and the origin can even be taken on the boundary of the material.

If the tangent at the point O on the boundary of a material be given by $\theta = -\alpha$ and $\theta = \pi-\alpha$, then considering the equilibrium of the material within a radius r of O, the forces on the boundary APB clearly have a resultant passing through O $(= X+iY)$. The forces on the boundary BOA consist of a finite distribution of boundary tractions over the arc AOB and a possible isolated force F $(= Re^{i\beta})$ at O. Hence from the equilibrium of the material in the circuit $APBOA$ we find

$$F+\text{force resultant } F' \text{ of boundary stresses over } AOB+\int_{-\alpha}^{\pi-\alpha}\breve{rr}e^{i\theta}r\,d\theta = 0,$$

i.e.
$$F+F'+\int_{-\alpha}^{\pi-\alpha}\tfrac{1}{2}(A/z+\bar{A}/\bar{z})z\,d\theta = 0,$$

or
$$F+F'+\tfrac{1}{2}\int_{-\alpha}^{\pi-\alpha}(A+\bar{A}e^{2i\theta})\,d\theta = 0.$$

This is independent of r, and therefore if $r \to 0$, $F' \to 0$, and $F+\frac{1}{2}\pi A = 0$, whence

$$A = -2F/\pi. \tag{4.2}$$

Hence the complex potentials for such a force F on the boundary at the origin O must (from 4.1) contain the terms

$$\Omega(z) = -\frac{2F}{\pi}\log z, \qquad \omega(z) = \frac{2\overline{F}}{\pi} z \log z. \tag{4.3}$$

5. Change of origin

The displacement D is given either by

$$8\mu D = \kappa\Omega(z) - z\overline{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}),$$

or by

$$8\mu D = \kappa\Omega_1(z_1) - z_1\overline{\Omega}_1'(\bar{z}_1) - \bar{\omega}_1'(\bar{z}_1),$$

where we have the two origins (1) $z = 0$, and (2) $z_1 = 0$ given by $z = c$, i.e. $z_1 = z - c$. Therefore

$$\kappa\Omega(z) - z\overline{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}) = \kappa\Omega_1(z_1) - z\overline{\Omega}_1(\bar{z}_1) - \{\bar{\omega}_1'(\bar{z}_1) - c\overline{\Omega}_1'(\bar{z}_1)\},$$

so that

$$\Omega(z) = \Omega_1(z_1), \qquad \omega(z) = \omega_1(z_1) - \bar{c}\Omega_1(z_1), \tag{5.1}$$

and these are the formulae for the stresses for a change of origin.

Therefore, for an isolated complex force F at a point $z = z_r$, putting $z - z_r = \zeta$,

$$\Omega_1(\zeta) = -\frac{2F}{\pi}\log\zeta, \qquad \omega_1(\zeta) = \frac{2\overline{F}}{\pi}\zeta\log\zeta,$$

so that

$$\Omega(z) = -\frac{2F}{\pi}\log\zeta, \tag{5.2}$$

and

$$\omega(z) = \frac{2\overline{F}}{\pi}(z - z_r)\log\zeta - \bar{z}_r\left(-\frac{2F}{\pi}\log\zeta\right)$$

$$= \frac{2\overline{F}}{\pi} z\log\zeta + \frac{2}{\pi}(F\bar{z}_r - \overline{F}z_r)\log\zeta. \tag{5.3}$$

Now

$$F\bar{z}_r = (X+iY)(x_r - iy_r) = Xx_r + Yy_r + i(Yx_r - Xy_r),$$

so that $F\bar{z}_r - \overline{F}z_r = 2iG$, where G is the moment of the force F at $z = z_r$ about the origin, and therefore

$$\omega(z) = \frac{2\overline{F}}{\pi} z\log\zeta + \frac{4iG}{\pi}\log\zeta. \tag{5.4}$$

6. Circular hole in an infinite plate under an isolated force on the boundary

For a complex force $F = Re^{i\beta}$ at the point A_1 ($z = a_1$, $a_1 = ae^{i\alpha}$) on the boundary of a circular hole in an infinite plate, equations (5.2) and (5.3) give the complex potentials

$$\Omega(z) = -\frac{2F}{\pi}\log(z - a_1) \tag{6.1}$$

and
$$\omega(z) = \frac{2\overline{F}}{\pi} z \log(z-a_1) + \frac{2}{\pi}(F\bar{a}_1 - \overline{F}a_1)\log(z-a_1), \tag{6.2}$$

and these give the requisite force at A_1.

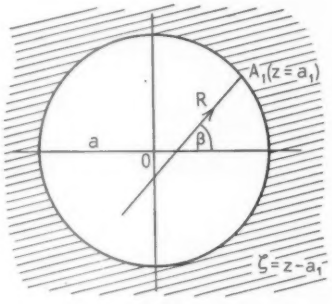

FIG. 3.

The stress resultants round the circle at infinity must balance the complex force F at A_1, hence the logarithmic part of the complex potentials must contain terms obtained from (3.5) and (3.6) by changing the sign of F and G $[= \frac{1}{2}i(F\bar{a}_1 - \overline{F}a_1)]$, i.e.

$$\Omega(z) = -\frac{2F\log z}{\pi(\kappa+1)} \tag{6.3}$$

and
$$\omega(z) = \frac{2\kappa\overline{F}z\log z}{\pi(\kappa+1)} + \frac{1}{\pi}(F\bar{a}_1 - \overline{F}a_1)\log z. \tag{6.4}$$

Hence the terms $\Omega(z) = A\log z$ and $\omega(z) = Bz\log z + C\log z$ must be added to the solution (6.1) and (6.2) so as to give the form (6.3) and (6.4).

If $\zeta = z - a_1$, then $\zeta/z = 1 - a_1/z$, and so (6.1) and (6.2) can be written

$$\Omega(z) = -\frac{2F}{\pi}\log z - \frac{2F}{\pi}\log\left(1-\frac{a_1}{z}\right)$$

and

$$\omega(z) = \frac{2\overline{F}}{\pi} z\log z + \frac{2}{\pi}(F\bar{a}_1 - \overline{F}a_1)\log z + \frac{2\overline{F}}{\pi} z\log\left(1-\frac{a_1}{z}\right) + \\ + \frac{2}{\pi}(F\bar{a}_1 - \overline{F}a_1)\log\left(1-\frac{a_1}{z}\right),$$

the terms in $\zeta/z = 1 - a_1/z$ vanishing at infinity. Hence, comparing the joint solution with (6.3) and (6.4)

$$A - \frac{2F}{\pi} = -\frac{2F}{\pi(\kappa+1)}, \qquad B + \frac{2\overline{F}}{\pi} = \frac{2\kappa\overline{F}}{\pi(\kappa+1)},$$

$$C + \frac{2}{\pi}(F\bar{a}_1 - \overline{F}a_1) = \frac{1}{\pi}(F\bar{a}_1 - \overline{F}a_1),$$

or $$A=\frac{2F\kappa}{\pi(\kappa+1)},\qquad B=-\frac{2\overline{F}}{\pi(\kappa+1)},\qquad C=-\frac{1}{\pi}(F\bar{a}_1-\overline{F}a_1),$$

and so the joint solution which has the correct form at infinity and gives the isolated force at the boundary is

$$\Omega=-\frac{2F}{\pi}\log\zeta+\frac{2F\kappa\log z}{\pi(\kappa+1)}+\Omega_0,$$

$$\omega=\frac{2\overline{F}}{\pi}z\log\zeta+\frac{2}{\pi}(F\bar{a}_1-\overline{F}a_1)\log\zeta-\frac{2\overline{F}z\log z}{\pi(\kappa+1)}-\frac{1}{\pi}(F\bar{a}_1-\overline{F}a_1)\log z+\omega_0,$$

where Ω_0 and ω_0 are expansible in negative powers of z, and are chosen to satisfy the boundary condition

$$\overline{\Omega}(\bar{z})+\bar{z}\Omega'(z)+\omega'(z)=k$$

round the boundary $z\bar{z}=a^2=a_1\bar{a}_1$. Now $\zeta=z-a_1$, therefore

$$\bar{\zeta}=\bar{z}-\bar{a}_1=\bar{a}_1(a/z-1)=-\bar{a}_1\zeta/z,$$

so that the boundary condition becomes

$$-\frac{2\overline{F}}{\pi}\log\left(-\frac{\bar{a}_1\zeta}{z}\right)+\frac{2\overline{F}\kappa\log(a^2/z)}{\pi(\kappa+1)}+\bar{z}\left(-\frac{2F}{\pi\zeta}+\frac{2F\kappa}{\pi(\kappa+1)z}\right)+$$
$$+\frac{2\overline{F}}{\pi}\left(\log\zeta+\frac{z}{\zeta}\right)+\frac{2}{\pi\zeta}(F\bar{a}_1-\overline{F}a_1)-\frac{2\overline{F}(\log z+1)}{\pi(\kappa+1)}-\frac{1}{\pi z}(F\bar{a}_1-\overline{F}a_1)+$$
$$+\overline{\Omega}_0(\bar{z})+\bar{z}\Omega_0'(z)+\omega_0'(z)=k$$

and as $\bar{\zeta}=-\bar{a}_1\zeta/z$, and

$$\bar{z}/\zeta=a^2/(\zeta z)=-(a^2/a_1)(1/z-1/\zeta)=\bar{a}_1(1/\zeta-1/z),$$

the above condition can be written

$$\frac{1}{\pi}\Big[2\overline{F}\log\zeta(1-1)+2\overline{F}\log z\left(1-\frac{\kappa}{\kappa+1}-\frac{1}{\kappa+1}\right)+\frac{4\overline{F}\kappa}{(\kappa+1)\log a}-$$
$$-2\overline{F}\log(-\bar{a}_1)-\frac{2\overline{F}}{\kappa+1}-2F\bar{a}_1\left(\frac{1}{\zeta}-\frac{1}{z}\right)+\frac{2F\kappa a^2}{(\kappa+1)z^2}+$$
$$+2\overline{F}(\zeta+a_1)/\zeta+2(F\bar{a}_1-\overline{F}a_1)/\zeta-(F\bar{a}_1-\overline{F}a_1)/z\Big]+$$
$$+\overline{\Omega}_0(\bar{z})+\bar{z}\Omega_0'(z)+\omega_0'(z)=k.$$

The constant terms $\log\zeta$, $\log z$, and the terms in $1/\zeta$ cancel out and the constant terms together with the constant k can be replaced by a new constant k', giving

$$\overline{\Omega}_0(\bar{z})+\bar{z}\Omega_0'(z)+\omega_0'(z)+\frac{F\bar{a}_1+\overline{F}a_1}{\pi z}+\frac{2F\kappa a^2}{\pi(\kappa+1)z^2}=k';$$

this can be satisfied by taking $\Omega_0(z)=0$ and

$$\omega_0(z)=-\frac{1}{\pi}(F\bar{a}_1+\overline{F}a_1)\log z+\frac{2F\kappa a^2}{\pi(\kappa+1)z},$$

as this gives finite stresses and displacements at infinity, and, since the coefficient of $\log z$ is real, $Cy\omega(z)$ is a pure imaginary, and so the moment condition is not affected.

Combining our solutions, we have

$$\Omega(z) = -\frac{2F}{\pi}\log(z-a_1)+\frac{2F\kappa\log z}{\pi(\kappa+1)}, \tag{6.5}$$

$$\omega(z) = \frac{2\overline{F}}{\pi}z\log(z-a_1)+\frac{2}{\pi}(F\bar{a}_1-\overline{F}a_1)\log(z-a_1)- \\ -\frac{2\overline{F}z\log z}{\pi(\kappa+1)}-\frac{2}{\pi}F\bar{a}_1\log z+\frac{2F\kappa a^2}{\pi(\kappa+1)z}. \tag{6.6}$$

For a number of forces F_r at $z = a_r$ round the hole, which is otherwise free from applied force, these complex potentials become

$$\Omega(z) = -\frac{2}{\pi}\sum F_r\log z_r+\frac{2\kappa\log z}{\pi(\kappa+1)}\sum F_r, \tag{6.7}$$

$$\omega(z) = \frac{2z}{\pi}\sum \overline{F}_r\log z_r+\frac{2}{\pi}\sum (F_r\bar{a}_r-\overline{F}_r a_r)\log z_r- \\ -\frac{2z\log z}{\pi(\kappa+1)}\sum \overline{F}_r-\frac{2\log z}{\pi}\sum F_r\bar{a}_r+\frac{2\kappa a^2}{\pi(\kappa+1)z}\sum F_r, \tag{6.8}$$

the terms containing $\sum F_r$ disappearing when the forces are in equilibrium.

Equal and opposite forces at the end of a chord

If there are two forces $F_1 = iR$ at $z = a_1$, and $F_2 = -iR$ at $z = a_2 = \bar{a}_1$, the complex potentials reduce to

$$\Omega = \frac{2iR}{\pi}\log\left(\frac{z-\bar{a}_1}{z-a_1}\right),$$

and as $\quad \sum F_r\bar{a}_r = iR\bar{a}_1-iR\bar{a}_1 = 0, \qquad \omega = \dfrac{2iRz}{\pi}\log\left(\dfrac{z-\bar{a}_1}{z-a_1}\right).$

Now $2(\breve{nn}+\breve{ss}) = \Omega'(z)+\overline{\Omega}'(\bar{z})$, and as $\breve{nn} = 0$ round the boundary

$$z\bar{z} = a^2 = a_1\bar{a}_1,$$

the peripheral stress $\breve{ss}$ is given by $2\breve{ss} = \Omega'(z)+\overline{\Omega}'(a^2/z)$, i.e.

$$\breve{ss} = (iR/\pi)[(\bar{a}_1-a_1)/(z^2-a^2-a_1z-\bar{a}_1z)]+\text{conjugate},$$

and if $a_1 = ae^{i\alpha}$, $z = ae^{i\theta}$ round the boundary, this becomes

$$\breve{ss} = \frac{2R\sin\alpha\sin\theta}{\pi a(\cos\theta-\cos\alpha)}.$$

It will be noticed that the peripheral stress is infinite at the point of application of the isolated forces, except in the particular case $\alpha = \frac{1}{2}\pi$, when $\breve{ss} = 2R/(\pi a) =$ constant round the hole.

7. Elliptic hole in an infinite plate under an isolated force on the boundary

The space outside an ellipse in the z-plane is conformally mapped on the outside of the unit circle in a complex σ-plane by the transformation

$$z = c(\sigma+\lambda\sigma^{-1}). \tag{7.1}$$

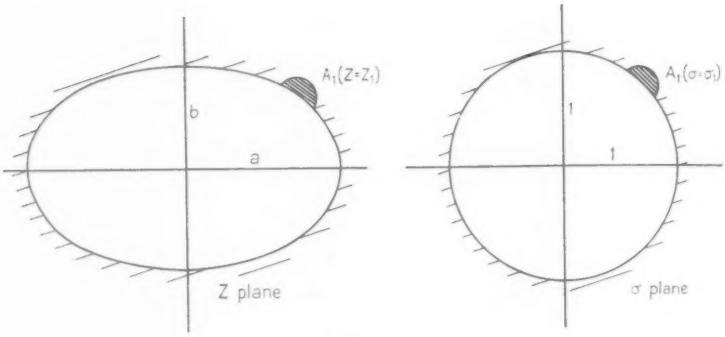

FIG. 4.

We write $\zeta = \sigma-\sigma_1$ and suppose that a complex force $F = Re^{i\beta}$ acts at the point A_1 ($z = z_1$, $\sigma = \sigma_1$, or $\zeta = 0$) on the boundary of the hole. Considering first the complex potentials which give the isolated force at A_1 in the z-plane, we take

$$\Omega_1 = A\log\zeta, \qquad \omega_1 = Bz\log\zeta+C\log\zeta, \tag{7.2}$$

where the constants A, B, C will be chosen to give the required force at A_1. This force together with the stresses round a small shaded region is in equilibrium, so that if the stress resultants along this arc are $X+iY$ and M, then

$$F+X+iY = 0$$

and
$$\tfrac{1}{2}i(\bar{F}z_1-F\bar{z}_1)+M = 0$$

where
$$X+iY = -\tfrac{1}{4}i\{\Omega(z)+z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})\}_A^B$$

and
$$M = \tfrac{1}{4}\text{ real part of }\{\omega(z)-z\omega'(z)-z\bar{z}\Omega'(z)\}_A^B,$$

and as we are only going round a point semicircle,

$$[\Omega]_A^B = \tfrac{1}{2}Cy\Omega, \quad \text{giving} \quad F = \tfrac{1}{4}\pi(\bar{B}-A). \tag{7.3}$$

Also
$$\tfrac{1}{2}i(\bar{F}z_1-F\bar{z}_1)+\tfrac{1}{8}\text{ real part of }(2\pi iC) = 0$$

so that if
$$C = C_1+iC_2, \quad \text{then} \quad iC_2 = (2/\pi)(F\bar{z}_1-\bar{F}z_1). \tag{7.4}$$

As $|\sigma| \to \infty$, $\log z \to \log \sigma$, and therefore at infinity the form of the complex potentials is

$$\Omega(z) = -\frac{2F\log z}{\pi(\kappa+1)}, \quad \text{i.e.} \quad \Omega(z) = -\frac{2F\log\sigma}{\pi(\kappa+1)}, \tag{7.5}$$

$$\omega(z) = \frac{2\kappa\overline{F}z\log z}{\pi(\kappa+1)} + \frac{1}{\pi}(F\bar{z}_1 - \overline{F}z_1)\log z,$$

i.e.
$$\omega(z) = \frac{2\kappa\overline{F}z\log\sigma}{\pi(\kappa+1)} + \frac{1}{\pi}(F\bar{z}_1 - \overline{F}z_1)\log\sigma, \tag{7.6}$$

since these will give, round the infinite circle, force and couple resultants which balance those of F at A_1. Therefore, taking* $\Omega_2 = D\log\sigma$, and $\omega_2 = Ez\log\sigma + G\log\sigma$, the combinations at infinity will be of the form

$$\Omega = (A+D)\log\sigma, \qquad \omega = (B+E)z\log\sigma + (C+G)\log\sigma$$

so agreeing with (7.5) and (7.6) if

$$A+D = -\frac{2F}{\pi(\kappa+1)}, \quad B+E = \frac{2\kappa\overline{F}}{\pi(\kappa+1)}, \quad i(C_2+G_2) = \frac{1}{\pi}(F\bar{z}_1 - \overline{F}z_1), \tag{7.7}$$

where
$$C = C_1 + iC_2 \quad \text{and} \quad G = G_1 + iG_2.$$

The complex potentials

$$\Omega = A\log\zeta + D\log\sigma + \Omega_0(\sigma),$$

and
$$\omega = Bz\log\zeta + C\log\zeta + Ez\log\sigma + G\log\sigma + \omega_0(\sigma)$$

are therefore taken, where Ω_0 and ω_0 must be developed in negative powers of σ and any rigid body displacements, i.e. terms of the type

$$\Omega = \alpha + i\beta z \quad (\beta \text{ real}), \qquad \omega = \gamma + \delta z,$$

can be neglected. The boundary condition

$$\overline{\Omega}(\sigma^{-1}) + \bar{z}(\sigma^{-1})\, d\Omega/dz + d\omega/dz = \text{constant} = -k,$$

or
$$z'(\sigma)\{\overline{\Omega}(\sigma^{-1}) + k\} + \bar{z}(\sigma^{-1})\Omega'(\sigma) + \omega'(\sigma) = 0 \tag{7.8}$$

has to be satisfied. Substituting for Ω and ω this reduces to

$$z'(\sigma)\{\overline{\Omega}_0(\sigma^{-1}) + \bar{A}\log\zeta - (\bar{A}+\bar{D})\log\sigma + k' + B\log\zeta + E\log\sigma\} +$$
$$+ \bar{z}(\sigma^{-1})\{\Omega_0'(\sigma) + A/\zeta + D/\sigma\} + \omega_0'(\sigma) + z(B/\zeta + E/\sigma) + C/\zeta + G/\sigma = 0,$$

where $k' = k + \bar{A}\log(-\bar{\sigma}_1) = \text{constant}$, and choosing

$$B + \bar{A} = 0 \quad \text{and} \quad E - (\bar{A}+\bar{D}) = 0 \tag{7.9}$$

the rigid body displacements in this equation can be eliminated; thence solving equations (7.3), (7.7), and (7.9) we find

$$A = -\frac{2F}{\pi}, \qquad B = \frac{2\overline{F}}{\pi}, \qquad D = \frac{2F\kappa}{\pi(\kappa+1)}, \qquad E = -\frac{2\overline{F}}{\pi(\kappa+1)}. \tag{7.10}$$

* Here, D must not be confused with the same symbol in section 1.

The boundary condition then reduces to

$$\omega_0'(\sigma)+C/\zeta+G/\sigma+c(\sigma+\lambda\sigma^{-1})(B/\zeta+E/\sigma)+\{\overline{\Omega_0'(\sigma^{-1})}+k'\}z'(\sigma)+$$
$$+\{\Omega_0'(\sigma)+A/\zeta+D/\sigma\}c(\sigma^{-1}+\lambda\sigma)=0,$$

and writing $\sigma/\zeta=-\bar{k}'$ so that $\Omega_0'(\sigma)=0$ and putting $B=-\bar{A}$, this further reduces to

$$\omega_0'(\sigma)+c\{\lambda(A+D)+(E-\bar{A})\}+c(D+\lambda E)\sigma^{-2}+$$
$$+(C+A\bar{z}_1-\bar{A}z_1)/\zeta+\{G\sigma_1-c(A-\lambda\bar{A})\}/(\sigma\sigma_1)=0.$$

The terms in ζ^{-1} and σ^{-1} in this equation can be made to disappear by choosing

$$G\sigma_1=c(A-\lambda\bar{A}) \tag{7.11}$$

and

$$C=\bar{A}z_1-A\bar{z}_1=2(F\bar{z}_1-\bar{F}z_1)/\pi \tag{7.12}$$

so that C is a pure imaginary ($\equiv iC_2$) and this is the value given in (7.4). Also from equation (7.7) $iG_2=-(F\bar{z}_1-\bar{F}z_1)/\pi$ which can be shown to be consistent with (7.11). Hence,

$$G=2\lambda c(F\sigma_1+\bar{F}\bar{\sigma}_1)/\pi-2F\bar{z}_1/\pi \tag{7.13}$$

and the boundary condition is satisfied by taking

$$\omega_0(\sigma)=-c\{\lambda(A+D)+(E-\bar{A})\}\sigma+c(D+\lambda E)\sigma^{-1}$$

to which may be added the rigid body displacement

so that
$$\omega_0(\sigma)=c\{\lambda(A+D)+(E-A)\}(\sigma+\lambda\sigma^{-1}),$$
$$\omega_0(\sigma)=c\{D+\lambda E+\lambda(E-\bar{A})+\lambda^2(A+D)\}\sigma^{-1}$$
$$=-\frac{2c}{\sigma\pi(\kappa+1)}\{F(\kappa-\lambda^2)+\bar{F}\lambda(\kappa-1)\}. \tag{7.14}$$

Hence the final solution for the problem of a single isolated force on the boundary of an elliptic hole in an infinite plate is

$$\Omega=-\frac{2F}{\pi}\log\zeta+\frac{2F\kappa\log\sigma}{\pi(\kappa+1)}\quad(\zeta=\sigma-\sigma_1), \tag{7.15}$$

$$\omega=\frac{2\bar{F}}{\pi}z\log\zeta+\frac{2}{\pi}(F\bar{z}_1-\bar{F}z_1)\log\zeta-\frac{2\bar{F}z\log\sigma}{\pi(\kappa+1)}+$$
$$+\left\{\frac{2\lambda c}{\pi}(F\sigma_1+\bar{F}\bar{\sigma}_1)-\frac{2F\bar{z}_1}{\pi}\right\}\log\sigma+\frac{2c}{\sigma\pi(\kappa+1)}\{F(\kappa-\lambda^2)+\bar{F}\lambda(\kappa-1)\}. \tag{7.16}$$

A number of forces in equilibrium

For a number of such complex forces F_r at $z = z_r$ (corresponding to $\sigma = \sigma_r$), these potentials become (writing $\zeta_r = \sigma - \sigma_r$)

$$\Omega = -\frac{2}{\pi}\sum F_r \log \zeta_r + \frac{2\kappa \log \sigma}{\pi(\kappa+1)}\sum F_r, \tag{7.17}$$

$$\omega = \frac{2z}{\pi}\sum \overline{F}_r \log \zeta_r + \frac{2}{\pi}\sum (F_r \bar{z}_r - \overline{F}_r z_r)\log \zeta_r - \frac{2z \log \sigma}{\pi(\kappa+1)}\sum \overline{F}_r +$$
$$+\left\{\frac{2\lambda c}{\pi}\sum (F_r \sigma_r + \overline{F}_r \bar{\sigma}_r) - \frac{2}{\pi}\sum F_r \bar{z}_r\right\}\log \sigma +$$
$$+\frac{2c}{\sigma\pi(\kappa+1)}\left\{(\kappa-\lambda^2)\sum F_r + \lambda(\kappa-1)\sum \overline{F}_r\right\}. \tag{7.18}$$

If, however, the forces are in equilibrium, then $\sum F_r = 0$ and $\sum F_r \bar{z}_r$ is real, and the complex potentials become

$$\Omega = -\frac{2}{\pi}\sum F_r \log \zeta_r, \tag{7.19}$$

$$\omega = \frac{2z}{\pi}\sum \overline{F}_r \log \zeta_r + \frac{2}{\pi}\sum (F_r \bar{z}_r - \overline{F}_r z_r)\log \zeta_r +$$
$$+\frac{2}{\pi}\log \sigma\{\lambda c \sum (\overline{F}_r \bar{\sigma}_r + F_r \sigma_r) - \sum F_r \bar{z}_r\}. \tag{7.20}$$

Equal and opposite forces at the ends of the minor axis

If there are two forces, $F_1 = iY$ and $F_2 = -iY$, at the points $z_1 = ib$ and $z_2 = -ib$, then

$$\sum (F_r \bar{z}_r - \overline{F}_r z_r) = 0 \quad \text{and} \quad \sum (F_r \sigma_r + \overline{F}_r \bar{\sigma}_r) = -4Y,$$

so that
$$\Omega = -\frac{2iY}{\pi}\log\left(\frac{\sigma - i}{\sigma + i}\right),$$

and
$$\omega = -\frac{2iYz}{\pi}\log\left(\frac{\sigma - i}{\sigma + i}\right) - \frac{4aY}{\pi}\log \sigma.$$

Then as $2(\breve{n}n + \breve{s}s) = \Omega'(\sigma)/z'(\sigma) + \overline{\Omega}'(\bar{\sigma})/\bar{z}'(\bar{\sigma})$, and $\breve{n}n$ is zero round the boundary, we get on simplification,

$$(\breve{s}s)_{\text{boundary}} = \frac{2Y(1-\lambda)}{\pi c(1+\lambda^2-2\lambda\cos 2\eta)},$$

where $\sigma = \cos\eta + i\sin\eta$. Also if a and b are the semi-axes of the ellipse and p is the perpendicular from the origin on to the tangent at the point whose eccentric angle is η the above result can be written in the alternative form

$$\breve{s}s = \frac{2Yb}{\pi(a^2\sin^2\eta + b^2\cos^2\eta)} = \frac{2Yp^2}{\pi a^2 b}.$$

This is a maximum at the end of the major axis where it has the value $2Y/\pi b$, and putting $a = b$ or $\lambda = 0$ in the above results, $\breve{ss} = 2Y/\pi a$, agreeing with the previous result for the circular hole. This result is the same as that derived by Green (**2**) and can be written in the form $(\breve{ss}.\pi a/Y) = 2a/b$ which has the values 2, 4·33, and 8·16 for $b/a = 1$, 0·462, and 0·245 respectively, agreeing with results otherwise calculated by Symonds (**3**). See Fig. 6, where $h = b/a$.

Equal and opposite forces at the ends of a diameter

In this case there are two forces F_1 at σ_1, and $-F_1$ at σ_2 ($\sigma_2 = -\sigma_1$), where F_1 is complex. Substituting in the above results we find

$$\Omega = -\frac{2F}{\pi}\log\left(\frac{\sigma-\sigma_1}{\sigma+\sigma_1}\right),$$

giving

$$\breve{ss} = \frac{2R[\lambda\sin(\beta+\eta)-\sin(\beta-\eta)]}{\pi c(1+\lambda^2-2\lambda\cos 2\eta)\sin(\eta-\alpha)},$$

where $F = Re^{i\beta}$ and $\sigma_1 = e^{i\alpha}$. When $\alpha = \beta = \frac{1}{2}\pi$ this reduces to the previous result for forces at the ends of the minor axis. See Fig. 7, drawn for $\beta = 45°$.

Equal and opposite forces at the ends of a chord (i) *parallel to the minor axis*, (ii) *parallel to the major axis*

(i) The two forces are iY at $\sigma = \sigma_1$ and $-iY$ at $\sigma = \sigma_2 = \bar{\sigma}_1$ giving

$$\Omega = -\frac{2iY}{\pi}\log\left(\frac{\sigma-\sigma_1}{\sigma-\bar{\sigma}_1}\right),$$

and writing $\sigma = e^{i\eta}$, $\sigma_1 = e^{i\alpha}$, then

$$\breve{ss} = \frac{2Y[\lambda\sin 2\eta\sin\eta\sin\alpha+\sin\alpha\cos\eta]}{\pi c(\cos\eta-\cos\alpha)(1+\lambda^2-2\lambda\cos 2\eta)}.$$

(ii) The two forces are Y at $\sigma = \sigma_1$ and $-Y$ at $\sigma = \bar{\sigma}_1$, giving

$$\breve{ss} = \frac{2Y[\cos\eta\sin\alpha(1-\lambda\cos 2\eta)-\lambda\sin\eta\cos\alpha\sin 2\eta]}{\pi c(\cos\eta-\cos\alpha)(1+\lambda^2-2\lambda\cos 2\eta)},$$

and when $\lambda = 0$, both these results reduce to that for the circular hole, i.e.

$$\frac{2Y\sin\alpha\cos\eta}{\pi c(\cos\eta-\cos\alpha)}.$$

8. Curvilinear polygonal hole in an infinite plate under isolated forces on the boundary

The conformal transformation, $z = c\sigma(1+\lambda\sigma^{-n})$ transforms the outside of a curvilinear polygonal boundary (Fig. 5) into the unit circle (**4**), where $\lambda < 1/(n-1)$.

We assume a complex force $F = Re^{i\beta}$ to act on the boundary of the hole

at the point A_1 ($z = z_1$), corresponding to the point $\sigma = \sigma_1$ on the unit circle in the σ-plane, where $\sigma - \sigma_1 = \zeta$ and $\sigma_1 = e^{i\alpha}$.

Also $\sigma = e^{i\eta}$ round the boundary of the hole, and we call η the eccentric angle of a point on this boundary by analogy with the eccentric angle of the ellipse (which is the special case $n = 2$).

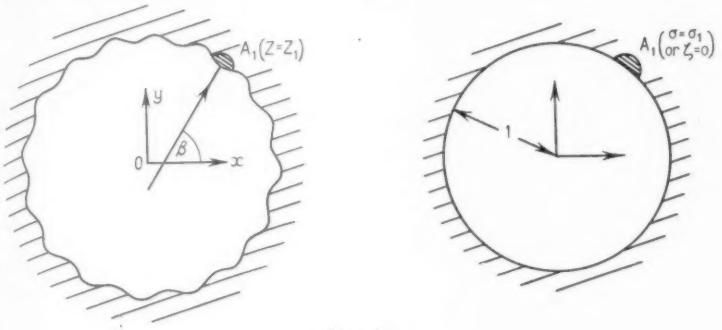

FIG. 5.

Proceeding as in the case of the elliptic hole, complex potentials

$$\Omega_1 = A \log \zeta, \qquad \omega_1 = Bz \log \zeta + C \log \zeta \tag{8.1}$$

are first taken and values for the constants A, B, C are chosen such that these potentials give the required complex force F at $z = z_1$, i.e.

$$F = \tfrac{1}{4}\pi(\overline{B} - A), \qquad iC_2 = \frac{2}{\pi}(F\bar{z}_1 - \overline{F}z_1). \tag{8.2}$$

Secondly potentials

$$\Omega_2 = D \log \sigma, \qquad \omega_2 = Ez \log \sigma + G \log \sigma \tag{8.3}$$

are taken so that the combinations $\Omega_1 + \Omega_2$ and $\omega_1 + \omega_2$ give the correct form at infinity and so provide the necessary balancing force and couple for the equilibrium of the plate, i.e.

$$\Omega = -\frac{2F \log z}{\pi(\kappa+1)},$$

$$\omega = \frac{2\kappa \overline{F} z \log z}{\pi(\kappa+1)} + \frac{2i}{\pi}(M + iM^*) \log z,$$

where M is the real part of $iF\bar{z}_1 = \frac{1}{2}i(F\bar{z}_1 - \overline{F}z_1)$ and as in equation (7.7)

$$A + D = -\frac{2F}{\pi(\kappa+1)}, \qquad B + E = \frac{2\kappa\overline{F}}{\pi(\kappa+1)}, \qquad i(C_2 + G_2) = \frac{1}{\pi}(F\bar{z}_1 - \overline{F}z_1),$$

and

$$iG_2 = -\frac{1}{\pi}(F\bar{z}_1 - \overline{F}z_1). \tag{8.4}$$

Potentials $\Omega = \Omega_0 + \Omega_1 + \Omega_2$ and $\omega = \omega_0 + \omega_1 + \omega_2$ are now taken, where

Ω_0 and ω_0 are expansible in negative powers of σ only, so that the conditions at infinity are unaltered, and Ω_0, ω_0 are chosen to satisfy the boundary condition round the hole, i.e.

$$z'(\sigma)\{\overline{\Omega}(\sigma^{-1})+k\}+\bar{z}(\sigma^{-1})\Omega'(\sigma)+\omega'(\sigma)=0 \quad \text{round} \quad \sigma\bar{\sigma}=1.$$

Now, $$\zeta=\sigma\left(1-\frac{\sigma_1}{\sigma}\right) \quad \text{so that} \quad \bar{\zeta}=-\frac{\sigma^{-1}}{\sigma_1\zeta},$$

and the boundary condition reduces to

$$z'(\sigma)\{\overline{\Omega}_0(\sigma^{-1})+k'+\bar{A}\log\zeta-(\bar{A}+\bar{D})\log\sigma\}+\bar{z}(\sigma^{-1})\{\Omega_0'(\sigma)+A/\zeta+D/\sigma\}+$$
$$+z'(\sigma)\{B\log\zeta+E\log\sigma\}+\omega_0'(\sigma)+C/\zeta+G/\sigma+z(\sigma)(B/\zeta+E/\sigma)=0.$$

Choosing $B+\bar{A}=0$ and $E-(\bar{A}+\bar{D})=0$, the logarithmic terms disappear, and these equations together with (8.4) can be solved for A, B, D, and E, whence the boundary condition further reduces to

$$z'(\sigma)\{\overline{\Omega}_0(\sigma^{-1})+k'\}+\bar{z}(\sigma^{-1})\Omega_0'(\sigma)+\omega_0'(\sigma)+f(\sigma)=0,$$

where

$$f(\sigma)=\frac{C}{\zeta}+\frac{G}{\sigma}+cE(1+\lambda\sigma^{-n})-c\bar{E}\kappa(\sigma^{-2}+\lambda\sigma^{n-2})+$$
$$+cA\left(\frac{1}{\sigma\zeta}+\frac{\lambda\sigma^{n-1}}{\zeta}\right)-c\bar{A}\left(\frac{\sigma}{\zeta}+\frac{\lambda}{\zeta\sigma^{n-1}}\right). \quad (8.5)$$

The terms in $1/\zeta$ and $1/\sigma$, which would only arise from the logarithmic terms already allowed for, have to be removed and $f(\sigma)$ expanded in powers of σ.

Now

$$\frac{1}{\zeta\sigma}=\frac{1}{\sigma_1}\left(\frac{1}{\zeta}-\frac{1}{\sigma}\right), \qquad \frac{\sigma}{\zeta}=1+\frac{\sigma_1}{\zeta},$$
$$\frac{\sigma^{n-1}}{\zeta}=\sigma^{n-2}+\sigma_1\sigma^{n-3}+\sigma_1^2\sigma^{n-4}+\ldots+\sigma_1^{n-2}+\frac{\sigma_1^{n-1}}{\zeta},$$

and

$$\frac{1}{\zeta\sigma^{n-1}}=\frac{1}{\sigma^{n-2}}\frac{1}{\sigma_1}\left(\frac{1}{\zeta}-\frac{1}{\sigma}\right)=\frac{1}{\sigma_1\sigma^{n-3}}\frac{1}{\sigma_1}\left(\frac{1}{\zeta}-\frac{1}{\sigma}\right)-\frac{1}{\sigma_1\sigma^{n-1}}$$
$$=\ldots=\frac{1}{\sigma_1^{n-1}}\left(\frac{1}{\zeta}-\frac{1}{\sigma}\right)-\frac{1}{\sigma_1^{n-2}}\frac{1}{\sigma^2}-\frac{1}{\sigma_1^{n-3}}\frac{1}{\sigma^3}-\ldots-\frac{1}{\sigma_1^2}\frac{1}{\sigma^{n-2}}-\frac{1}{\sigma_1}\frac{1}{\sigma^{n-1}}$$

so that $f(\sigma)$ can be written

$$\zeta^{-1}[C+Ac(\sigma_1^{-1}+\lambda\sigma_1^{n-1})-\bar{A}c(\sigma_1+\lambda\sigma_1^{-n+1})]+\sigma^{-1}[G-c(A\sigma_1^{-1}-\lambda\bar{A}\sigma_1^{-n+1})]+$$
$$+c(E-\bar{A})+c\lambda A\sigma_1^{n-2}+c\sigma^{-2}[-\bar{E}\kappa+\lambda\bar{A}\sigma_1^{-n+2}]+c\sigma^{n-2}\lambda(A-\bar{E}\kappa)+$$
$$+c\lambda A[\sigma_1\sigma^{n-3}+\sigma_1^2\sigma^{n-4}+\ldots+\sigma_1^{n-3}\sigma]+$$
$$+c\lambda\bar{A}[\sigma_1^{-n+3}\sigma^{-3}+\ldots+\sigma_1^{-2}\sigma^{-n+2}+\sigma_1^{-1}\sigma^{-n+1}];$$

also C and G are chosen such that

$$C=C_1+iC_2=\bar{A}z_1-A\bar{z}_1=2(F\bar{z}_1-\bar{F}z_1)/\pi,$$

and therefore comparing with (8.2)

$$C = iC_2 = 2(F\bar{z}_1 - \bar{F}z_1)/\pi \tag{8.6}$$

and

$$G = c(A\sigma_1^{-1} - \lambda\bar{A}\sigma_1^{-n+1}) = -\lambda c(\sigma_1^{-n+1}\bar{A} + A\sigma_1^{n-1}) + cA(\sigma_1^{-1} + \lambda\sigma_1^{n-1})$$

or

$$G = G_1 + iG_2 = 2\lambda c(F\sigma_1^{n-1} + \bar{F}\bar{\sigma}_1^{n-1})/\pi - 2F\bar{z}_1/\pi,$$

therefore

$$iG_2 = (\bar{F}z_1 - F\bar{z}_1)/\pi \tag{8.7}$$

agreeing with (8.4).

All the constants A, B, C, D, E, and G are therefore determined and $f(\sigma)$ reduces to

$$\begin{aligned} f(\sigma) = c[E - \bar{A} + \lambda A\sigma_1^{n-2}] + \lambda c(A - \bar{E}\kappa)\sigma^{n-2} - c(\bar{E}\kappa - \lambda\bar{A}\sigma_1^{-n+2})\sigma^{-2} + \\ + c\lambda A[\sigma_1^{n-3}\sigma + \sigma_1^{n-4}\sigma^2 + \ldots + \sigma_1^2\sigma^{n-4} + \sigma_1\sigma^{n-3}] + \\ + c\lambda\bar{A}[\sigma_1^{-n+3}\sigma^{-3} + \sigma_1^{-n+4}\sigma^{-4} + \ldots + \sigma_1^{-2}\sigma^{-n+2} + \sigma_1^{-1}\sigma^{-n+1}]. \end{aligned} \tag{8.8}$$

Now $\omega = \alpha z$, where α is a constant, gives a rigid body displacement which can be ignored, so that we can add the solution

$$\omega'(\sigma) = -(E - \bar{A} + \lambda A\sigma_1^{n-2})z'(\sigma) = -c(E - \bar{A} + \lambda A\sigma_1^{n-2})\{1 - (n-1)\lambda\sigma^{-n}\}$$

whence (8.8) becomes

$$\begin{aligned} f(\sigma) = \lambda c(A - \bar{E}\kappa)\sigma^{n-2} + c\lambda A[\sigma_1\sigma^{n-3} + \sigma_1^2\sigma^{n-4} + \ldots + \sigma_1^{n-4}\sigma^2 + \sigma_1^{n-3}\sigma] - \\ - c\sigma^{-2}(\bar{E}\kappa - \lambda\bar{A}\bar{\sigma}_1^{n+2}) + c\lambda\bar{A}[\sigma_1^{-n+3}\sigma^{-3} + \sigma_1^{-n+4}\sigma^{-4} + \ldots + \\ + \sigma_1^{-2}\sigma^{-n+2} + \sigma_1^{-1}\sigma^{-n+1}] + c(n-1)\lambda\sigma^{-n}(E - \bar{A} + \lambda A\sigma_1^{n-2}). \end{aligned}$$

Now taking

$$\Omega_0 = \sum_1^{n-2} A_r\sigma^{-r}, \qquad \omega_0 = c\sum_1^{n-1} B_r\sigma^{-r} + cB_{-1}\sigma, \tag{8.9}$$

(8.5) becomes

$$\begin{aligned} z'(\sigma)\{\overline{\Omega}_0(\sigma^{-1}) + k'\} + \bar{z}(\sigma^{-1})\Omega_0'(\sigma) + \omega_0'(\sigma) \\ = c[1 - \lambda(n-1)\sigma^{-n}]\Big\{k' + \sum_1^{n-2}\bar{A}_r\sigma^r\Big\} - c[\sigma^{-1} + \lambda\sigma^{n-1}]\sum_1^{n-2} rA_r\sigma^{-r-1} + \\ + cB_{-1} - c\sum_1^{n-1} rB_r\sigma^{-r-1} \\ = c\Big\{k' + B_{-1} - (n-2)\lambda A_{n-2} + \sum_1^{n-3}[\bar{A}_r - \lambda(n-2-r)A_{n-2-r}]\sigma^r - \\ - \sum_3^{n-1}\{\lambda(n-1)\bar{A}_{n-r} + (r-2)A_{r-2} - (r-1)B_{r-1}\}\sigma^{-r} + \bar{A}_{n-2}\sigma^{n-2} - \\ - [B_1 + \lambda(n-1)\bar{A}_{n-2}]\sigma^{-2} - \\ - [\lambda(n-1)k' + (n-2)A_{n-2} + (n-1)B_{n-1}]\sigma^{-n}\Big\}, \end{aligned} \tag{8.10}$$

so that from (8.8) and (8.10) the boundary condition can be satisfied by taking

$$\lambda(A-\bar{E}\kappa)+\bar{A}_{n-2}=0,$$

$$\lambda A\sigma_1^{n-r-2}+\bar{A}_r-\lambda(n-2-r)A_{n-2-r}=0 \quad (1\leqslant r\leqslant n-3),$$

$$\kappa\bar{E}-\lambda\bar{A}\sigma_1^{-n+2}+B_1+\lambda(n-1)\bar{A}_{n-2}=0,$$

$$-\lambda\bar{A}\sigma_1^{-n+r}+\lambda(n-1)\bar{A}_{n-r}+(r-2)A_{r-2}-(r-1)B_{r-1}=0 \quad (3\leqslant r\leqslant n-1),$$

$$-(n-1)\lambda[E-\bar{A}+\lambda A\sigma_1^{n-2}]+\lambda(n-1)k'+(n-2)A_{n-2}+(n-1)B_{n-1}=0,$$

$$k'+B_{-1}-(n-2)\lambda A_{n-2}=0.$$

Putting $k'=0$ and solving these equations:

$$A_{n-2}=\frac{2\lambda\bar{F}}{\pi(\kappa+1)},$$

$$A_r=\frac{2[\lambda^2F(n-2-r)\sigma_1^r+\lambda\bar{F}\bar{\sigma}_1^{n-r-2}]}{\pi[1-\lambda^2r(n-r-2)]},$$

$$B_1=\frac{2F[\kappa-(n-1)\lambda^2]}{\pi(\kappa+1)}-\frac{2\lambda\bar{F}}{\pi}\sigma_1^{-n+2},$$

$$B_{-1}=\frac{2(n-2)\lambda^2\bar{F}}{\pi(\kappa+1)},$$

$$B_{n-1}=\frac{2\lambda\bar{F}}{\pi(\kappa+1)}\left(\kappa-\frac{n-2}{n-1}\right)-\frac{2\lambda^2F}{\pi}\sigma_1^{n-2},$$

$$rB_r=\frac{2\lambda\bar{F}}{\pi}\sigma_1^{-n+r+1}+\frac{2[\bar{F}\lambda(r-1)\bar{\sigma}_1^{n-r-1}\{1+\lambda^2(n-1)\}]}{\pi[1-\lambda^2(n-r-1)(r-1)]}+$$

$$+\frac{2[F\lambda^2\sigma_1^{r-1}\{(n-1)r-(r-1)^2\}]}{\pi[1-\lambda^2(n-r-1)(r-1)]} \quad (2\leqslant r\leqslant n-2).$$

The case $n=4$

As an example we consider the case of a curvilinear square (i.e. $n=4$) when there are isolated forces at the vertices given by

$$F=R \text{ at } \sigma=1, \quad iR \text{ at } \sigma=i, \quad -R \text{ at } \sigma=-1,$$

and

$$-iR \text{ at } \sigma=-i.$$

Then substituting in the above equations,

$$\Omega_1=-\frac{2R}{\pi}\log\left(\frac{\sigma-1}{\sigma+1}\right)-\frac{2iR}{\pi}\log\left(\frac{\sigma-i}{\sigma+i}\right),$$

$$\Omega_2=0, \qquad \Omega_0=\sum A_1\sigma^{-1}+\sum A_2\sigma^{-2},$$

where

$$A_1=\frac{2(\lambda^2F\sigma_1+\lambda\bar{F}\bar{\sigma}_1)}{\pi(1-\lambda^2)} \quad \text{and} \quad A_2=\frac{2\lambda\bar{F}}{\pi(\kappa+1)},$$

so that

$$\sum A_1=0, \quad \text{and} \quad \sum A_2=0,$$

therefore

$$\Omega=-\frac{2R}{\pi}\log\left(\frac{\sigma-1}{\sigma+1}\right)-\frac{2iR}{\pi}\log\left(\frac{\sigma-i}{\sigma+i}\right).$$

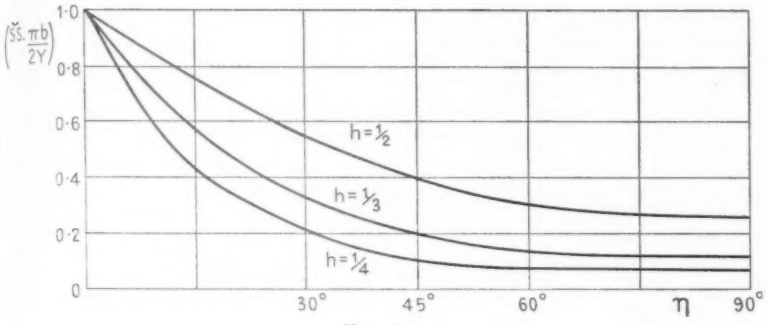

FIG. 6.

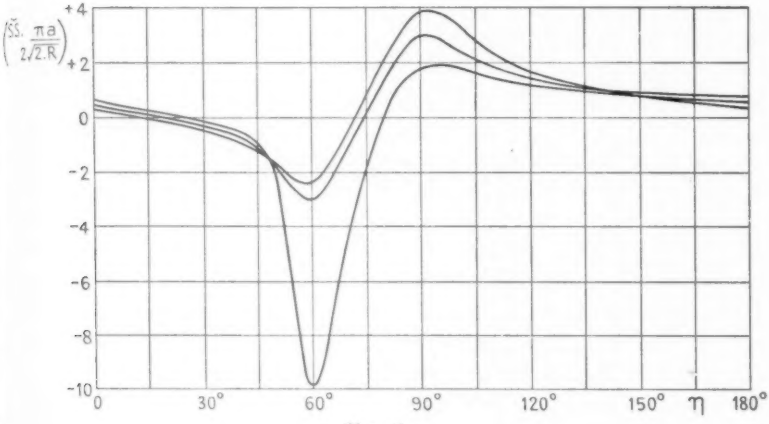

FIG. 7.

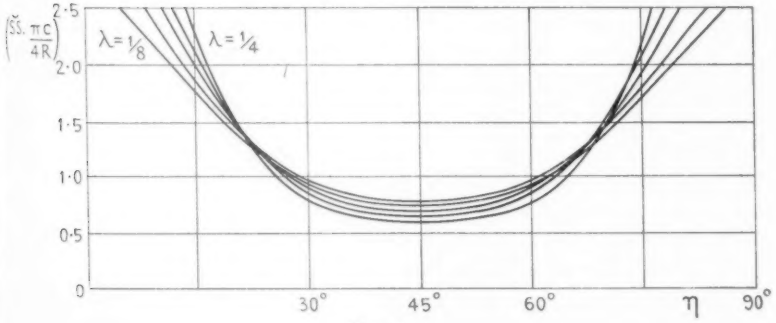

FIG. 8.

Also $z'(\sigma) = c(1-3\lambda\sigma^{-4})$, $\lambda < \frac{1}{3}$, so that $\breve{nn}$ being zero, the peripheral stress $\breve{ss}$ is given by:

$$2\breve{ss} = \Omega'(\sigma)/z'(\sigma)+\overline{\Omega}'(\bar{\sigma})/\bar{z}'(\bar{\sigma})$$

$$= \frac{4R(1+3\lambda)}{\pi c(1-6\lambda\cos 4\eta+9\lambda^2)}.$$

This is symmetrical about both axes and has the value $4R(1+3\lambda)/\pi c(1-3\lambda)^2$ at the points of application of the forces. See Fig. 8; $\lambda = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$.

I wish to express my thanks to Mr. A. C. Stevenson for suggesting these problems and for many helpful suggestions and criticisms in their solution.

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A NOTE ON THE HODOGRAPH TRANSFORMATION FOR THE TWO-DIMENSIONAL VORTEX FLOW OF AN INCOMPRESSIBLE FLUID

By S. GOLDSTEIN *and* M. J. LIGHTHILL

(*Department of Mathematics, The University, Manchester*)

[Received 11 October 1949]

SUMMARY

In the hodograph transformation of the two-dimensional vortex flow of an incompressible, inviscid fluid, branch lines arise, similar to those previously encountered in the theory of the irrotational supersonic flow of a gas. The result is of importance for the calculation of vortex flows with 'free' streamlines, and in particular for the calculation of jets with vorticity present. As a simple mathematical example the hodograph plane is described corresponding to the flow of a stream with uniform shear past a circular cylinder; the hodograph plane is a Riemann surface of six sheets.

1. Introduction

In Kirchhoff's classical solution (**1**) for the two-dimensional motion of an incompressible fluid in a jet issuing through an aperture in a plane wall, the motion was taken to be irrotational, and the same is true of recent extensions (**2**). In a real fluid, however, a boundary layer will be formed along the wall, and probably the most important effects will be due to the vorticity in the boundary layer and not to the viscosity of the fluid. W. T. Lord therefore began a study of two-dimensional vortex motion in jets. Solutions of such problems may have practical applications in the design of slots for sucking away a boundary layer. As a beginning, it was thought that a solution might be sought for a fluid possessing regions of constant vorticity, or, as an easier alternative, for the case of a single concentrated vortex sheet near the wall upstream of the aperture and passing into the jet. As for problems of irrotational motion in jets, the hodograph transformation would be used, the components, u, v, of the fluid velocity (or $\log q$ and ϑ, where q is the magnitude of the resultant fluid velocity and ϑ the angle the velocity makes with a fixed direction) being taken as independent variables. It was found, however, that the transformation from the physical to the hodograph plane is not a one-to-one transformation, and the hodograph plane is a Riemann surface. In the case of a concentrated vortex sheet the surface folds back over itself at a winding point (as in analytic-function theory), but with distributed vorticity it folds along a curve, on the whole of which the Jacobian of the transformation vanishes. These curves are 'branch lines' of the same type

as those previously encountered in the theory of the irrotational supersonic flow of a gas (**3**, **4**, **5**). Because of the occurrence of branch lines, it will not be a simple matter to use the hodograph transformation for the study of flows with vorticity, and problems with 'free' streamlines will be difficult. Of the cases previously mentioned, the problem of the jet with a small constant vorticity is manageable, and has been studied further by W. T. Lord; the case of a concentrated vortex sheet may also be manageable, and will be studied further.

If x and y are rectangular cartesian coordinates, and u and v the corresponding components of the fluid velocity, then for a two-dimensional motion, with vorticity ϵ, of an incompressible fluid, u and v satisfy the equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\epsilon.$$

When $\epsilon = 0$ any singularities of the transformation from the (x, y) plane to the (u, v) plane are isolated points; when ϵ is not zero, branch lines (on which $\partial(u, v)/\partial(x, y) = 0$) may occur for any value of ϵ, no matter how small.

In order to demonstrate the occurrence of branch lines in the hodograph transformation of the two-dimensional vortex flow of an incompressible inviscid fluid we work out in the next paragraph the hodograph plane corresponding to the flow of a stream with uniform shear past a circular cylinder. This is a simple mathematical example, of no physical interest in itself, intended purely to exhibit the kind of mathematical difficulty that may arise in using the hodograph transformation for other problems involving vortex motion.

In his investigation of the geometry of the hodograph transformation, Craggs (**5**) showed that branch lines, defined as lines along which the Jacobian of the transformation vanishes, cannot occur in irrotational subsonic flow, and it has not previously been pointed out that they do occur in vortex flows. Craggs also showed that a folding back of the hodograph plane is usual along such lines, and that in general branch lines are cusped in the hodograph plane; these statements are verified in the example below.

2. The flow of a stream with uniform shear past a circular cylinder

With rectangular Cartesian coordinates (x, y), let the undisturbed values of the velocity components u, v of a stream of inviscid incompressible fluid be $u = U-\epsilon y$, $v = 0$, where ϵ is a constant. Let the stream flow past the circular cylinder, $x^2+y^2 = 1$, of unit radius (Fig. 1). The motion being two-dimensional, the vorticity will be everywhere equal to ϵ, by

Helmholtz's theorems; the fluid being incompressible there will be a stream function ψ, such that $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, which satisfies $\nabla^2\psi = -\epsilon$,

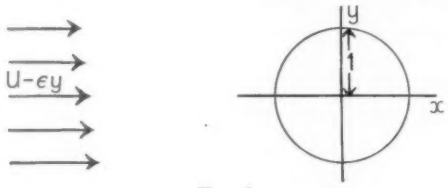

FIG. 1.

and which is constant on the cylinder. The solution giving the correct undisturbed motion is, with r and θ as polar coordinates,

$$\psi = Uy - \tfrac{1}{2}\epsilon y^2 - \frac{Uy}{x^2+y^2} + \tfrac{1}{4}\epsilon\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$= U\left(r - \frac{1}{r}\right)\sin\theta - \tfrac{1}{4}\epsilon r^2 + \tfrac{1}{4}\epsilon\left(r^2 - \frac{1}{r^2}\right)\cos 2\theta, \tag{1}$$

$$U - u = \frac{U}{r^2}\cos 2\theta + \epsilon r\sin\theta - \frac{\epsilon}{2r^3}\sin 3\theta, \tag{2}$$

$$-v = \frac{U}{r^2}\sin 2\theta + \frac{\epsilon}{2r^3}\cos 3\theta. \tag{3}$$

The branch lines are the lines along which $\partial(u,v)/\partial(x,y)$ vanishes. Now

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{r}\frac{\partial(u,v)}{\partial(r,\theta)} = -\frac{4U^2}{r^6} + \frac{2\epsilon U}{r^3}\sin 3\theta + \frac{6\epsilon U}{r^7}\sin\theta - \frac{9\epsilon^2}{4r^8} + \frac{3\epsilon^2}{2r^4}\cos 4\theta. \tag{4}$$

The curve $\partial(u,v)/\partial(x,y) = 0$ is a real curve. To proceed further, assume that ϵ is small; then r must be large on the curve whose equation is approximately

$$r^3\sin 3\theta = 2U/\epsilon, \tag{5}$$

(and on which $r \geqslant (2U/\epsilon)^{\frac{1}{3}}$). The curve has three branches, lying between the asymptotes $\theta = 0$ and $\theta = \pi/3$, $\theta = 2\pi/3$ and $\theta = \pi$, and $\theta = 4\pi/3$ and $\theta = 5\pi/3$, respectively (Fig. 2).

To the same approximation, the last terms on the right in (2) and (3) may be omitted.

In the hodograph plane take coordinates X, Y, defined by

$$X = U - u, \qquad Y = -v. \tag{6}$$

Then from (2), (3), and (5) the parametric equations of the branch line in the hodograph plane are, to the same approximation,

$$X = \epsilon\left(\frac{2U}{\epsilon\sin 3\theta}\right)^{\frac{1}{3}}(\sin\theta + \tfrac{1}{2}\cos 2\theta\sin 3\theta), \tag{7}$$

$$Y = \left(\frac{\epsilon\sin 3\theta}{2U}\right)^{\frac{2}{3}} U\sin 2\theta. \tag{8}$$

In $0 \leqslant \theta \leqslant \pi/3$, X goes from 0 to $+\infty$ and Y rises from zero and falls to zero again. In $2\pi/3 \leqslant \theta \leqslant \pi$, X goes from $+\infty$ to 0 and Y falls from zero and rises to zero again. In $4\pi/3 \leqslant \theta \leqslant 5\pi/3$, X goes from $-\infty$ back to $-\infty$, remaining always negative, and Y rises from zero, falls through zero

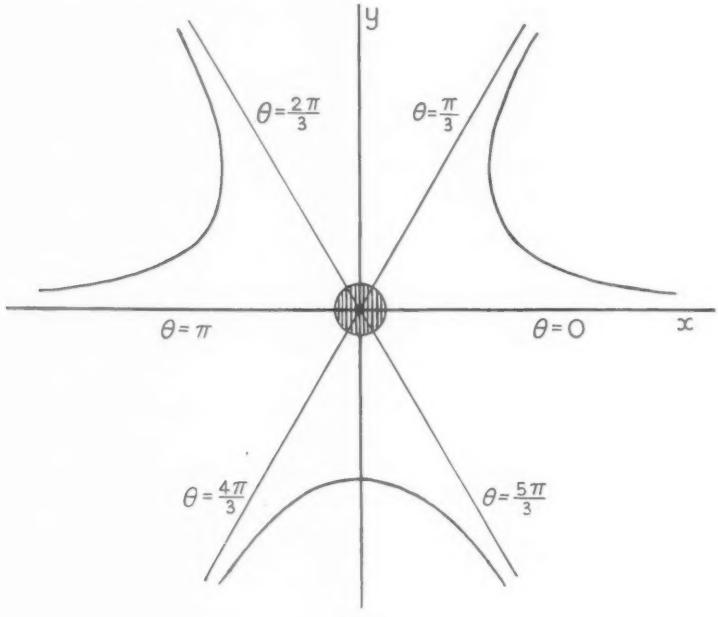

FIG. 2.

to a negative value, and rises to zero again. $dY/d\theta$ vanishes when $\sin 5\theta = 0$, the relevant solutions for θ inside the above ranges being

$$\theta = \frac{\pi}{5}, \frac{4\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}. \tag{9}$$

$dX/d\theta$ vanishes when $\cos 3\theta = 0$ or $\sin 5\theta = 0$. The zeros of $\sin 5\theta$ given in (9) are therefore zeros both of $dX/d\theta$ and of $dY/d\theta$, and are cusps. $X = Y = 0$ is also a cusp. X as a function of Y has stationary values at those zeros of $\cos 3\theta$ for which $\theta = \pi/6$, $5\pi/6$, and $3\pi/2$, and it is easily shown that the first two of these are maxima and the third a minimum. From these results the branch line in the hodograph plane may be sketched (Fig. 3). There are five cusps.

The mapping of the physical plane on to the hodograph plane may be studied by considering the behaviour of lines $\theta =$ constant, which (from

(6), and from (2) and (3) without the last terms on the right) are the cubics without points of inflexion in the hodograph plane whose equation is

$$X = Y \cot 2\theta + \epsilon \sin\theta \left(\frac{U \sin 2\theta}{Y}\right)^{\frac{1}{2}}. \tag{10}$$

The details of this investigation are omitted; the results are as follows.

The hodograph plane is a Riemann surface of six sheets. Three sheets, which we shall call sheets 1, 2, and 3, do not contain the cusped figure $EFHG$; the other three sheets—sheets 4, 5, and 6—do not contain the cusped figure $OABDC$.

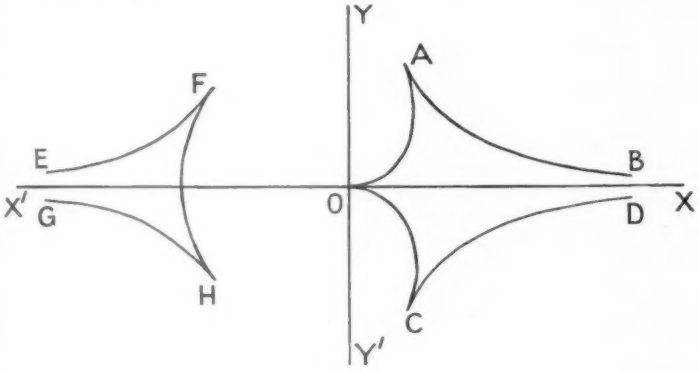

FIG. 3.

Sheet 1 is the whole plane, cut along AB and CD. Sheets 2 and 3 each consist only of the interior of $OABDC$, cut along OX. The right-hand edges of the cuts AB and CD on sheet 1 are joined to sheet 3. The left-hand edges of these cuts are joined to sheet 2; thus at these joins the surface folds back over itself. Sheets 2 and 3 are joined along OA and along OC; thus at these joins the surface again folds back over itself.

Sheet 4 is the whole plane, cut along $X'OX$, EF, and GH. The upper edge of the half-cut OX on sheet 4 is joined to the lower edge of the cut OX on sheet 3, and the lower edge of OX on sheet 4 is joined to the upper edge on sheet 3. Sheets 5 and 6 each consist only of the interior of $EFGH$, not cut at all. The left-hand edges of the cuts EF and GH on sheet 4 are joined to sheet 6. The right-hand edges of these cuts are joined to sheet 5; at these joins the surface folds back over itself. Sheets 5 and 6 are joined along FH, which is another fold.

The cuts OX on sheet 2 and $X'O$ on sheet 4 are impassable barriers on the Riemann surface. Neither corresponds to any point at all in the physical plane.

The part of the plane inside each cusped figure is covered four times by the Riemann surface. The rest of the plane is covered twice.

The complexity of the hodograph plane in this simple example is a warning of the difficulty that may be encountered in using the hodograph transformation to study flows with vorticity present.

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REFLECTION AT A LAMINAR BOUNDARY LAYER OF A WEAK STEADY DISTURBANCE TO A SUPERSONIC STREAM, NEGLECTING VISCOSITY AND HEAT CONDUCTION

By M. J. LIGHTHILL

(*Department of Mathematics, The University, Manchester*)

[Received 25 February 1950]

SUMMARY

A perfect gas, with constant specific heats and zero viscosity and thermal conductivity, is considered to flow in a steady two-dimensional pattern which is only slightly disturbed from a basic flow parallel to a solid plane wall. The Mach number of the basic flow varies continuously and monotonically from zero at the wall to a supersonic value at a distance δ from the wall, beyond which it takes this supersonic value uniformly. The disturbance is considered to be due to a weak stationary simple plane wave incident on the 'boundary layer' from outside. The streamline pattern and pressure coefficient distribution, resulting from a given incident wave, are found to depend only on the Mach number distribution in the basic flow (not on the individual velocity and temperature distributions). Any singularity in the incident wave is propagated along characteristics in the supersonic part of the layer (increasing in strength) and is reflected from the sonic line modified in character. Thus a discontinuous increase (in pressure or pressure gradient) is reflected as a positive logarithmic infinity of the quantity (which result of a linearized theory may correspond to a ridge in the real values). The upstream influence of the disturbance on the layer is shown (for all Mach number distributions) to be completely negligible at a distance of one layer thickness ahead of the point of incidence of the wave front on the outside of the layer. (It is explained in section 8 why this conclusion, in the limiting case of a discontinuous Mach number distribution considered by Tsien and Finston (**1**), is at variance with their results.) For this reason the theory must be completely inadequate as a theory of shock laminar boundary layer interaction, and the significance of this conclusion is discussed in section 1. Four radically different possible mechanisms for the interaction (of a more complicated nature) are suggested, none of which can be discarded in the light of the data known to the author.

Some readers may be assisted by the concise account in section 4 of the more useful results of Langer's theory (**2**) of differential equations with a large complex parameter.

1. Introduction. Reflection of shocks from laminar boundary layers

It is fully established (**3, 4**) that, in steady supersonic flow, a wall with laminar boundary layer attached behaves, for purposes of shock reflection, very differently from the solid boundaries of the ordinary inviscid theory. Some diffusion of a disturbance, in the process of reflection by the laminar

layer was, of course, to be expected. But the experimental results, that upstream influence of the shock extends a distance 'tens of boundary layer thicknesses' ahead of the point of incidence, and that a shock is reflected locally as an *expansion wave*, are somewhat surprising. Given those results, the others accord with previous concepts. The boundary layer thickness increases ahead of the shock (over and above any normal thickening by diffusion of shear) to a peak at the point of incidence, whence it declines (or at least ceases to rise). As would be expected, compression waves from the region of thickening are found to combine with compression waves from behind the expansion wave to annul the latter, and form two shocks (as envelopes of Mach lines) which join into one (the reflected shock of the ordinary theory) at a distance of order a hundred boundary layer thicknesses from the wall (**4**). But this effect can naturally be erased by convex wall curvature. The fundamental phenomenon is that the reflected wave consists of a rather sharp expansion in the midst of a band of compression, which band, especially upstream of the point of incidence, is remarkably wide.

Two radically opposite views as to the correct basis for an explanation are held; neither basis has yet been developed into a full theory. Lagerstrom, Cole, and Trilling (**5**), following Liepmann, Ashkenas, and Cole (**4**), see the problem as a purely supersonic one, bound up intimately with the action of viscosity near the base of the shock, whose non-zero thickness needs to be considered. They magnify the possibilities of 'interaction between longitudinal and transverse waves' in viscous compressible flow, citing as an instance the known flow pattern for supersonic flow past a blunt-nosed plate parallel to the stream, which, however, is usually regarded as fully explicable on adiabatic theory (except in so far as boundary layer separation may be involved for certain nose shapes). They construct a linearized theory of viscous gas flow without heat conduction, which replaces the wall boundary condition of zero velocity by one that the velocity take a constant value slightly less than that of the stream. This they propose to apply to the interaction problem.

On the other hand, Tsien and Finston (**1**), following Howarth (**6**), hold that the subsonic portion of the boundary layer is the chief factor governing the anomalous shock reflection, which is usually absent for a turbulent layer (when the reflected wave is only displaced by a distance of the order of the layer thickness, and is not altered in character) because for this the region of subsonic mean speed is usually microscopic. They expect viscosity and heat conduction to affect only slightly the local modification of the laminar layer by a shock, and this view has more reason behind it than might first appear. The air in the boundary layer is not 'more

viscous' than that outside; it merely has a greater vorticity and lower Mach number. Viscosity takes time to influence any fluid element—thus it causes the layer to grow only slowly at the higher Reynolds numbers—*except* inside a shock, but here the velocity gradient is quite extraordinarily large and, on the other hand, the overall change is uninfluenced by the magnitude of the viscosity. The shock thickness is small compared with the layer thickness except where its strength tends to zero on approaching the sonic line (more precisely, where the reciprocal of its strength is comparable with the Reynolds number based on layer thickness, i.e. with the square root of the Reynolds number based on surface length); on the other hand, here it is a less significant phenomenon.

The theory to be given here is a modification of that of Tsien and Finston, in the direction that they themselves suggest as likely to give greatest improvement. They represent the boundary layer as a region of uniform subsonic flow, thus replacing its vorticity by two concentrated vortex sheets, one at the interface with the supersonic main stream and one at the wall. Here the same work will be undertaken with a layer of finite (distributed) vorticity, giving continuous decrease of velocity from that of the main stream to zero at the wall. As in the work of Tsien and Finston the equations of plane, steady, adiabatic flow are used, and the squares of disturbances to the parallel flow neglected.

Now the results found below, relating to the distance of upstream influence of the shock, are in complete disagreement with the experiments, making this distance far too small, and much smaller than in Tsien and Finston's theory. Yet the model of the flow has undeniably been improved. This indicates a possible element of chance about the good agreement of Tsien and Finston's work in this respect, but does not necessarily annul the validity of their general concepts. Four possibilities exist:

(i) Lagerstrom, Cole, and Trilling are right in ignoring the subsonic layer and ascribing the results principally to viscous diffusion in the supersonic layer; *or*

(ii) the theory of this paper could be corrected, by taking viscosity into account after the manner of the theories of boundary layer stability, especially near the wall where the full viscous boundary condition could then be applied; *or*

(iii) viscosity is not important, and the present theory is correct for a really weak disturbance, but the pattern changes rapidly as the shock amplitude is increased, so that to predict the effect theoretically the full non-linear adiabatic equations would be needed;* *or*

* This view appears less improbable when the focusing of disturbances as they near the sonic line (section 6) is taken into account.

(iv) there is a bubble of dead air at the wall near the base of the shock (perhaps slightly in reversed flow) of considerable length but small breadth; this has been formed by repeated separation until the pressure distribution was such as to permit attachment of the main laminar layer (which is outside the bubble).

Hypothesis (iv) as well as (iii) would imply considerable variation in the distance of upstream influence with shock strength; the evidence is negative on this point but not conclusively so. If the same qualitative behaviour is found for very weak shocks as for stronger ones, then the fundamental explanation must be (i) or (ii), and the effects described in (iii) and (iv) can only modify the phenomenon gradually as the shock strength is increased. But this is not yet certain. The author proposes to investigate hypothesis (ii) mathematically, and also (unless stronger evidence against them becomes available) hypotheses (iii) and (iv), but decided first to publish the present investigation without delay as an essential datum to the understanding of this much discussed topic.

The evidence of the theory of this paper regarding the local reflection of the shock by the laminar layer may possibly be of value even though that relating to the distance of upstream influence is not. This evidence is that it is from the sonic line that the shock is reflected as a rather sharp disturbance, and that the latter is in the nature of a pressure *ridge*, such as might be produced if the boundary layer turned rapidly but continuously through an angle to produce a band of compression waves, and then turned instantaneously back in the opposite direction (giving a Prandtl–Meyer expansion). The latter at least is observed (see above). What compression waves there are ahead of it interact with the expansion further from the wall (this could not be reproduced by the linear theory) and help to annul it and form a shock as an envelope of characteristics.

Unless hypothesis (ii) is the correct one, the theory of this paper is at least of interest as elucidating the reflection of a *smooth* weak disturbance.

[*Note added in proof.* Experiments conducted by Mair and Bardsley at the Manchester University Fluid Motion Laboratory have now established that the conclusions of the present theory *are* correct for the reflection of a weak shock by a *turbulent* boundary layer. At a Mach number just less than 2, any shock, weak enough to produce a deflexion $\leqslant 5°$, is found to be reflected locally as a pressure ridge, shown on the Schlieren records as a black line immediately adjacent to (and ahead of) a white line; the latter (representing the expansion) appears equally strong where both emerge from the boundary layer, but becomes weaker further out by interaction with the compression

waves behind the ridge. There is no visible upstream influence or boundary layer separation; and thus all the predictions of the theory agree with observation in this range of strengths. (With stronger incident shocks the reflected shock and expansion wave lie further apart, and the reflected shock springs from upstream of the point of incidence, and is so strong that the layer separates.)

It is not perhaps surprising that the theoretical conclusions are unaffected by the turbulence in the boundary layer, as one might expect the reflection of a stationary shock to depend principally on the mean flow. The significant deduction appears, therefore, to be that the present theory is correct for velocity profiles typical of turbulent layers, but incorrect for those typical of laminar layers. Preliminary calculations, based on hypothesis (ii) above, support this conjecture, for they indicate that the low Mach number layer very near the wall can probably send disturbances far upstream by a mechanism involving viscous stresses; now this inner layer would presumably be of negligible thickness in the turbulent case—which might be expected to nullify the effect.]

2. Mathematical formulation of the idealized problem

Let a perfect gas, with constant adiabatic index γ and zero viscosity and thermal conductivity, flow in a steady two-dimensional pattern, which differs only by a small disturbance from a basic flow in the x-direction parallel to a solid plane wall $y = 0$. Let the pressure p have a uniform value P in the basic flow; and let the velocity vector (u, v) and the density ρ therein be functions $\{U(y), 0\}$ and $R(y)$ of y alone, which are continuous for all y and constant for $y \geqslant \delta$. Let $A(y) = \{\gamma P/R(y)\}^{\frac{1}{2}}$ be the value in the basic flow of the local velocity of sound $a = (\gamma p/\rho)^{\frac{1}{2}}$, and suppose that the function $M(y) = U(y)/A(y)$ is non-decreasing in $0 \leqslant y \leqslant \delta$ from a value $M(0) = 0$ (where $M'(0) > 0$) to a value $M(\delta) = M_1 > 1$. Accents are used to denote derivatives.

Then the equations of the disturbed flow, expressing constancy of entropy along a streamline and conservation of mass and momentum, are

$$\left.\begin{aligned} u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y} &= a^2\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right), & \frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) &= 0, \\ \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x} &= 0, & \rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y} &= 0. \end{aligned}\right\} \tag{1}$$

When linearized, on the assumption that $u-U(y)$, v, $\rho-R(y)$, $p-P$,

$a-A(y)$ are small, equations (1) become

$$\left.\begin{aligned} U(y)\frac{\partial p}{\partial x} &= A^2(y)\left(U(y)\frac{\partial \rho}{\partial x}+R'(y)v\right), \\ U(y)\frac{\partial \rho}{\partial x}+R(y)\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\{R(y)v\} &= 0, \\ R(y)U(y)\frac{\partial u}{\partial x}+R(y)U'(y)v+\frac{\partial p}{\partial x} &= 0, \\ R(y)U(y)\frac{\partial v}{\partial x}+\frac{\partial p}{\partial y} &= 0. \end{aligned}\right\} \quad (2$$

Now elimination of $\partial\rho/\partial x$ and $\partial u/\partial x$ from the first three of equations (2 gives

$$R(y)U'(y)v+\frac{\partial p}{\partial x} = U(y)\frac{\partial}{\partial y}\{R(y)v\}+\frac{U^2(y)}{A^2(y)}\frac{\partial p}{\partial x}-U(y)R'(y)v, \quad (3$$

or

$$\frac{\partial p}{\partial x}\{1-M^2(y)\} = R(y)U^2(y)\frac{\partial}{\partial y}\left(\frac{v}{U(y)}\right). \quad (4$$

But actually $R(y)U^2(y) = \gamma PM^2(y)$. This suggests that the transformations

$$\frac{p-P}{\gamma P} = \varpi, \qquad \frac{v}{U(y)} = \eta, \quad (5$$

be made (so that ϖ is a pressure coefficient and η is the approximate deflexion—i.e. angle made by a streamline with the x-axis); then equation (4) and the last of equations (2) become

$$\{1-M^2(y)\}\frac{\partial \varpi}{\partial x} = M^2(y)\frac{\partial \eta}{\partial y}, \qquad \frac{\partial \varpi}{\partial y} = -M^2(y)\frac{\partial \eta}{\partial x}. \quad (6$$

Thus the distribution of pressure coefficient ϖ, and the streamline pattern (which is deducible from the distribution of deflexion η), depend not on the separate velocity and temperature profiles in the layer, but only on the distribution of Mach number $M(y)$. The equation for ϖ alone is obtained, by eliminating η, as

$$\frac{\partial^2 \varpi}{\partial y^2}-\frac{2M'(y)}{M(y)}\frac{\partial \varpi}{\partial y}+\{1-M^2(y)\}\frac{\partial^2 \varpi}{\partial x^2} = 0. \quad (7$$

It is considerably simpler than that for η alone.

The cause of the disturbance will be taken to be a simple plane wave incident on the layer $0 < y < \delta$ from outside. In it the pressure coefficient ϖ must take the approximate form $f(x+\beta y)$, where $\beta = (M_1^2-1)^{\frac{1}{2}}$, by the

theory of small disturbances to a uniform supersonic stream. It is reasonable to suppose that if f is uniformly small ϖ must be uniformly small. The use of (7) is therefore justified unless the derivatives of ϖ are nevertheless not uniformly small; they probably are not if f possesses a discontinuity (as would be needed to represent a shock); however, it will be hoped that some information can be obtained by solving equation (7).

The reflected wave in $y > \delta$ must take the form $\varpi = g(x-\beta y)$, and the function g is principally what we want to find. Since

$$\varpi = f(x+\beta y)+g(x-\beta y) \quad \text{for} \quad y \geqslant \delta,$$

the boundary condition at $y = \delta$ is

$$\left[\beta\frac{\partial\varpi}{\partial x}+\frac{\partial\varpi}{\partial y}\right]_{y=\delta} = 2\beta f'(x+\beta\delta), \tag{8}$$

provided that the derivatives are continuous across $y = \delta$. But discontinuity of ϖ or of η across an approximate streamline like $y = \delta$ is impossible by their physical significance. Hence $\partial\varpi/\partial x$ and $\partial\eta/\partial x$ are continuous across $y = \delta$, and since $M(y)$ is continuous, equations (6) show that $\partial\varpi/\partial y$ and $\partial\eta/\partial y$ are also continuous. This argument has been phrased to avoid assuming $M'(y)$ continuous at $y = \delta$, which might be inconvenient when model velocity profiles are chosen.

To complete the formulation of the problem the boundary condition at the wall $y = 0$ appropriate to an inviscid fluid is needed. This is $v = 0$. Hence $M^2(y)\partial\eta/\partial y = 0$, so by (6) $\partial\varpi/\partial x = 0$. Since $\varpi = 0$ at infinity upstream this gives

$$(\varpi)_{y=0} = 0. \tag{9}$$

(It will be seen later that this with equation (7) implies that $\varpi = O(y^3)$ as $y \to 0$.)

It is the boundary condition (9) which may, in a sense, be the cause of the disagreement with experiment. Its modification would be the aim of investigation (ii) of section 1. But it is incontestable for an inviscid fluid, for which therefore equations (7) must be solved under conditions (8) and (9) to get the pressure field, whence the reflected wave will be derived by the equation

$$\left[\beta\frac{\partial\varpi}{\partial x}-\frac{\partial\varpi}{\partial y}\right]_{y=\delta} = 2\beta g'(x-\beta\delta). \tag{10}$$

3. Solution as a Fourier integral

If $f(x)$ can be expressed as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} e^{ikx}F(k)\,dk, \tag{11}$$

it is reasonable to seek solutions for ϖ and g in the form

$$\varpi(x,y) = \int_{-\infty}^{\infty} e^{ikx}\Pi(k,y)\,dk, \qquad g(x) = \int_{-\infty}^{\infty} e^{ikx}G(k)\,dk. \tag{12}$$

Then equation (7) becomes an ordinary differential equation for Π,

$$\frac{d^2\Pi}{dy^2} - \frac{2M'(y)}{M(y)}\frac{d\Pi}{dy} + k^2\{M^2(y)-1\}\Pi = 0, \tag{13}$$

and the boundary conditions (8) and (9) become

$$\Pi_y(k,\delta)+\beta ik\Pi(k,\delta) = 2\beta ike^{ik\beta\delta}F(k), \qquad \Pi(k,0) = 0. \tag{14}$$

Choose $Q(k,y)$ to be any solution of equation (13) with $Q(k,0) = 0$. Then by (14)

$$\Pi(k,y) = 2\beta ike^{ik\beta\delta}F(k)\frac{Q(k,y)}{Q_y(k,\delta)+\beta ikQ(k,\delta)}. \tag{15}$$

Further, by equation (10),

$$\begin{aligned} G(k) &= [\beta ik\Pi(k,\delta)-\Pi_y(k,\delta)]/2\beta ike^{-ik\beta\delta} \\ &= e^{2ik\beta\delta}F(k)\frac{\beta ikQ(k,\delta)-Q_y(k,\delta)}{\beta ikQ(k,\delta)+Q_y(k,\delta)}. \end{aligned} \tag{16}$$

Actually, on the assumption made in section 2 that $M'(0) > 0$, the point $y = 0$ is a regular singularity of equation (13), at which the exponents are 0 and 3. Therefore the simplest form of $Q(k,y)$, and that which will be adopted, satisfies

$$Q(k,y) \sim y^3 \quad \text{as} \quad y \to 0 \tag{17}$$

for all k. In terms of this solution of (13), which (17) defines uniquely, equations (15) and (16), with (12), provide a complete formal solution to the problem of section 1.

Two uses will be made of this formal solution. In section 5, by studying the asymptotic behaviour of $Q(k,y)$ as $|k| \to \infty$, those of $\Pi(k,y)$ and $G(k)$ will be found. These will be used in section 6 to find what singularities in ϖ and g are produced by a specific singularity, such as a discontinuity, in f. For it is well known how the singularities of a Fourier integral at finite points can be deduced from the asymptotic behaviour of the integrand. (This investigation also enables the convergence of the Fourier integrals (12) to be demonstrated.)

Secondly, in section 7, the zeros of the common denominator of (15) and (16) will be considered. Since the integrals (12) can each be expressed in terms of the residues of the integrand at poles with $\operatorname{im} k > 0$ for $x > 0$, and in terms of those at poles with $\operatorname{im} k < 0$ for $x < 0$, knowledge of the zeros of the said denominator with least $|\operatorname{im} k|$ will enable the asymptotic disturbance to the boundary layer for large $|x/\delta|$ to be found. In particular,

the zero $-i\kappa_1$, with numerically smallest negative imaginary part, will determine the asymptotic form (proportional to $e^{\kappa_1 x}$ as $x \to -\infty$) of the upstream disturbance.

4. Résumé of Langer's theory

The asymptotic behaviour of $Q(k, y)$ as $|k| \to \infty$ (which is of different form in the subsonic and supersonic regions) will be studied by the method first fully developed by Langer (and foreshadowed by Jeffreys and others). A brief statement of the relevant conclusions of this theory is first given, as these are somewhat hard to extract from Langer's original paper (**2**).

In the equation

$$\frac{d^2u}{dy^2} + p(y)\frac{du}{dy} + \{k^2 q(y) + r(y)\}u = 0, \tag{18}$$

let $p(y)$, $q(y)$, and $r(y)$ be twice differentiable* in the interval $y_0 \leqslant y \leqslant y_2$. Let $q(y) < 0$ in $y_0 \leqslant y < y_1$ and $q(y) > 0$ in $y_1 < y \leqslant y_2$, while $q'(y_1) > 0$. Then two asymptotic solutions (valid as $|k| \to \infty$ uniformly in $y_0 \leqslant y \leqslant y_2$ for all $\arg k$) are

$$\left.\begin{aligned} u_1(y) &= \exp\left(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\right)|s|^{\frac{1}{2}}|q|^{-\frac{1}{4}}(ks)^{\frac{1}{2}}J_{-\frac{1}{3}}(ks)\sqrt{(2\pi)}, \\ u_2(y) &= \exp\left(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\right)|s|^{\frac{1}{2}}|q|^{-\frac{1}{4}}\operatorname{sgn}(y-y_1)(ks)^{-\frac{1}{2}}J_{\frac{1}{3}}(ks)\sqrt{(2\pi)}, \end{aligned}\right\} \tag{19}$$

where
$$s = \int_{y_1}^{y} \sqrt{q}\,dy \sim [q'(y_1)]^{\frac{1}{2}}\tfrac{2}{3}(y-y_1)^{\frac{3}{2}} \quad \text{as} \quad y \to y_1. \tag{20}$$

Here s is imaginary for $y < y_1$, but $(ks)^{-\nu}J_\nu(ks)$, involving only even powers of s, is real, and also independent of which square root of q is used in the integral for s; hence both u_1 and u_2 have these properties. It is also differentiable with respect to y, even at $y = y_1$. But also $|s|^{\frac{1}{2}}|q|^{-\frac{1}{4}}$ and $|s|^{\frac{1}{2}}|q|^{-\frac{1}{4}}\operatorname{sgn}(y-y_1)$ are differentiable even at $y = y_1$, and as $y \to y_1$

$$\frac{|s|^{\frac{1}{2}}}{|q|^{\frac{1}{4}}} \sim \frac{[q'(y_1)]^{\frac{1}{12}}(\frac{2}{3})^{\frac{1}{2}}|y-y_1|^{\frac{1}{4}}}{[q'(y_1)]^{\frac{1}{4}}|y-y_1|^{\frac{1}{4}}} = \left(\frac{2}{3q'(y_1)}\right)^{\frac{1}{6}}, \qquad \frac{|s|^{\frac{1}{2}}}{|q|^{\frac{1}{4}}}\operatorname{sgn}(y-y_1) \sim \left(\frac{2}{3q'(y_1)}\right)^{\frac{5}{6}} q(y), \tag{21}$$

so that

$$\left.\begin{aligned} u_1(y_1) &= \left(\frac{2}{3q'(y_1)}\right)^{\frac{1}{6}} \frac{2^{\frac{1}{3}}\sqrt{(2\pi)}}{(-\frac{1}{3})!}, \qquad u_1'(y_1) = 0, \qquad u_2(y_1) = 0, \\ u_2'(y_1) &= (\tfrac{2}{3})^{\frac{1}{3}}\{q'(y_1)\}^{\frac{1}{3}} \frac{2^{-\frac{1}{3}}\sqrt{(2\pi)}}{(\frac{1}{3})!}. \end{aligned}\right\} \tag{22}$$

* The theory may be extended to the case when at a finite number of exceptional points (but not at y_1) the functions are merely continuous, not twice differentiable.

When $|ks|$ is large (e.g. for fixed $y \neq y_1$ as $|k| \to \infty$) the functions $z^{-\nu}J_\nu(z)\sqrt{(2\pi)}$ in (19) may be replaced by their universal asymptotic formula as $|z| \to \infty$, namely

$$z^{-\nu}J_\nu(z)\sqrt{(2\pi)} \sim \frac{e^{iz}}{(iz)^{\nu+\frac{1}{2}}} + \frac{e^{-iz}}{(-iz)^{\nu+\frac{1}{2}}}, \tag{23}$$

which is valid for all $\arg z$ provided that the principal value of both $(\nu+\frac{1}{2})$th powers be taken. Hence, as $|ks| \to \infty$,

$$\left.\begin{aligned} u_1(y) &\sim \exp\left(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\right)\frac{|s|^{\frac{1}{4}}}{|q|^{\frac{1}{4}}}\left[\frac{e^{iks}}{(iks)^{\frac{1}{4}}} + \frac{e^{-iks}}{(-iks)^{\frac{1}{4}}}\right], \\ u_2(y) &\sim \exp\left(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\right)\frac{|s|^{\frac{3}{4}}\operatorname{sgn}(y-y_1)}{|q|^{\frac{1}{4}}}\left[\frac{e^{iks}}{(iks)^{\frac{3}{4}}} + \frac{e^{-iks}}{(-iks)^{\frac{3}{4}}}\right]. \end{aligned}\right\} \tag{24}$$

Thus for large $|ks|$ the general solution is some combination of solutions of the form

$$\exp\left(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\right)|q|^{-\frac{1}{4}}e^{\pm iks}. \tag{25}$$

For real k these are oscillatory when $y > y_1$ and exponential when $y < y_1$. If a special solution is determined by some boundary condition in either of these sub-intervals, then the asymptotic form of this boundary condition as $|k| \to \infty$ will determine what combination of the forms (25) is required. To permit continuation into the other sub-interval (or into the intermediate range where $|k|$ but not $|ks|$ is large) expression of the resulting asymptotic solution as a combination of the forms (24) is needed (being careful to take principal values of the $(\frac{1}{4})$th and $(\frac{3}{4})$th powers inside the square brackets); when this is done the special solution has been expressed in terms of u_1 and u_2 and so knowledge about it for large $|k|$ is complete.

For example in section 5 a solution will be found to have the asymptotic form as $|k| \to \infty$, for $y < y_1$,

$$u \sim \exp\left(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\right)|q|^{-\frac{1}{4}}\cosh[k(\sigma-|s|)], \tag{26}$$

where $\sigma > 0$. If it is asymptotically $f_1(k)u_1(y)+f_2(k)u_2(y)$, uniformly as $|k| \to \infty$, then by (24)

$$f_1(k)\left[\frac{e^{k|s|}}{k^{\frac{1}{4}}} + \frac{e^{-k|s|}}{(-k)^{\frac{1}{4}}}\right] - f_2(k)\left[\frac{e^{k|s|}}{k^{\frac{3}{4}}} + \frac{e^{-k|s|}}{(-k)^{\frac{3}{4}}}\right] \sim \cosh[k(\sigma-|s|)], \tag{27}$$

for all $\arg k$. The solution is easily verified* to be

$$f_1(k) \sim \frac{1}{2\sqrt{3}}\{e^{k\sigma}k^{\frac{1}{6}}+e^{-k\sigma}(-k)^{\frac{1}{6}}\}, \qquad f_2(k) \sim \frac{1}{2\sqrt{3}}\{e^{k\sigma}k^{\frac{1}{6}}+e^{-k\sigma}(-k)^{\frac{1}{6}}\}. \quad (28)$$

The behaviour of u as $|k| \to \infty$ for $y > y_1$ is then deduced, from (24) with positive s, as

$$u \sim \exp\Big(-\tfrac{1}{2}\int_{y_1}^{y} p\,dy\Big)q^{-\frac{1}{4}}[e^{iks}\cosh(k\sigma-\tfrac{1}{4}\pi i)+e^{-iks}\cosh(k\sigma+\tfrac{1}{4}\pi i)], \quad (29)$$

for all $\arg k$.

5. Asymptotic behaviour of Q, G, and Π

Equation (13) does not satisfy the conditions of the theory of section 4 in the whole interval $0 \leqslant y \leqslant \delta$, since $p(y) = -2M'(y)/M(y)$ is not differentiable (not, indeed, finite) at $y = 0$. But if ϵ is a small positive number the conditions *are* satisfied in $\epsilon \leqslant y \leqslant \delta$. The y_1 of section 4 must be the solution of $M(y_1) = 1$. *Thus y_1 is the thickness of the subsonic layer.* And when $\epsilon \leqslant y < y_1$, for example, the asymptotic form of $Q(k, y)$ as $|k| \to \infty$ must be some combination of the forms (25), which for equation (13) become

$$M(y)[1-M^2(y)]^{-\frac{1}{4}}e^{\pm k's}, \qquad s = \int_{y_1}^{y}[M^2(y)-1]^{\frac{1}{2}}\,dy. \quad (30)$$

But to deduce what combination is required, from the boundary condition (17) for $Q(k, y)$, information about the asymptotic behaviour right down to $y = 0$ is needed. This can be obtained from the fact that the second-order equation for d^2Q/dy^2 does satisfy the conditions of Langer's theory near $y = 0$ (as will not be surprising to the reader when he considers that in the general solution to (13) the values of $d^2\Pi/dy^2$ and $d^3\Pi/dy^3$ are arbitrary, indicating an ordinary point of the equation for the second derivative). Therefore the combination of the asymptotic forms (30) required to represent $Q(k, y)$ must represent it correctly down to $y = 0$ *as far as coefficients of powers of y higher than the first are concerned.*

Thus, by (17), the coefficient of y^3 in the asymptotic form must tend to 1, and that of y^2 must tend to 0, as $|k| \to \infty$. This fixes it as

$$M(y)[1-M^2(y)]^{-\frac{1}{4}}\cosh\Big(k\int_0^y[1-M^2(y)]^{\frac{1}{2}}\,dy\Big)\Big/[\tfrac{1}{2}k^2M'(0)], \quad (31)$$

so that, as $|k| \to \infty$ for $\epsilon \leqslant y < y_1$ and all $\arg k$,

$$\tfrac{1}{2}k^2M'(0)Q(k, y) \sim M(y)[1-M^2(y)]^{-\frac{1}{4}}\cosh[k(\sigma-|s|)], \quad (32)$$

* Noticing that $e^{\pm\pi i/6}-e^{\pm 5\pi i/6} = \sqrt{3}$, whether the upper or lower signs be taken.

as in equation (26) above, where

$$\sigma = \int_0^{y_1} [1-M^2(y)]^{\frac{1}{2}}\, dy = |s(0)|. \tag{33}$$

The transition to the asymptotic behaviour of Q in the supersonic region has been carried out in section 4. By (29), as $|k| \to \infty$ for $y_1 < y \leqslant \delta$ and all $\arg k$,

$$\tfrac{1}{2}k^2 M'(0) Q(k,y)$$

$$\sim M(y)[M^2(y)-1]^{-\frac{1}{4}}[e^{iks}\cosh(k\sigma-\tfrac{1}{4}\pi i)+e^{-iks}\cosh(k\sigma+\tfrac{1}{4}\pi i)] \tag{34}$$

$$= M(y)[M^2(y)-1]^{-\frac{1}{4}}[e^{k\sigma}\cos(ks-\tfrac{1}{4}\pi)+e^{-k\sigma}\cos(ks+\tfrac{1}{4}\pi)]. \tag{35}$$

Forms (34) and (35) are both useful. For real k the latter is split up into a part important only for positive k and another important only for negative k.

The common denominator of (15) and (16) can now be asymptotically estimated. Since $\beta = (M_1^2-1)^{\frac{1}{2}} = (ds/dy)_{y=\delta}$, and putting $(s)_{y=\delta} = S$, equation (34) gives

$$\tfrac{1}{2}k^2 M'(0)[Q_y(k,\delta)+\beta ikQ(k,\delta)] \sim M_1\beta^{-\frac{1}{2}}2\beta ike^{ikS}\cosh(k\sigma-\tfrac{1}{4}\pi i). \tag{36}$$

Here similar terms, involving $e^{\pm ikS}$ but no factor k, have been discarded; this is valid (since the terms retained are of higher order) except when the imaginary part $\operatorname{im} k$ of k is large and positive, when e^{-ikS} cannot be neglected in comparison with ke^{ikS}. Similarly for the numerator of (16), when $\operatorname{im} k$ is not large and negative,

$$\tfrac{1}{2}k^2 M'(0)[-Q_y(k,\delta)+\beta ikQ(k,\delta)] \sim M_1\beta^{-\frac{1}{2}}2\beta ike^{-ikS}\cosh(k\sigma+\tfrac{1}{4}\pi i). \tag{37}$$

Thus for bounded $\operatorname{im} k$, and in particular for real k, as $|k| \to \infty$, (16) becomes

$$G(k) \sim e^{2ik\beta\delta}F(k)e^{-2ikS}\frac{\cosh(k\sigma+\frac{1}{4}\pi i)}{\cosh(k\sigma-\frac{1}{4}\pi i)} \sim e^{2ik(\beta\delta-S)}F(k)i\operatorname{sgn} k. \tag{38}$$

As for $\Pi(k,y)$, when $|k| \to \infty$ for bounded $\operatorname{im} k$ and $\epsilon \leqslant y < y_1$, by (15), (32), and (36),

$$\Pi(k,y) \sim e^{ik(\beta\delta-S)}F(k)M_1^{-1}\beta^{\frac{1}{2}}M(y)[1-M^2(y)]^{-\frac{1}{4}}\frac{\cosh[k(\sigma-|s|)]}{\cosh(k\sigma-\frac{1}{4}\pi i)}$$

$$= O\{F(k)e^{-|ks|}\}, \tag{39}$$

so that, if the integrand in (11) is bounded at $k = \pm\infty$, then the integral (12) for $\varpi(x,y)$ is convergent by a large exponential factor, indicating regularity as a function of x.

But as $|k| \to \infty$ for real k and $y_1 < y \leqslant \delta$, by (15), (34), and (36),

$$\Pi(k,y) \sim F(k) M_1^{-1} \beta^{\frac{1}{2}} M(y) [M^2(y)-1]^{-\frac{1}{4}} [e^{ik(\beta\delta+s-S)} + e^{ik(\beta\delta-s-S)} i \operatorname{sgn} k]. \quad (40)$$

For $y = y_1$ or, more generally speaking, whenever $|k|$ is large but $|ks|$ is not, neither asymptotic form of Π is valid. Then it is necessary, in estimating $\frac{1}{2}k^2 M'(0) Q(k,y)$, to use its expression in terms of the functions (19) with the coefficients (28). In particular, by (22),

$$\begin{aligned} \tfrac{1}{2}k^2 M'(0) Q(k,y_1) &\sim f_1(k) u_1(y_1) \\ &\sim \frac{1}{2\sqrt{3}} \{e^{k\sigma} k^{\frac{1}{3}} + e^{-k\sigma}(-k)^{\frac{1}{3}}\} \left(\frac{2}{3.2M'(y_1)}\right)^{\frac{1}{6}} \frac{2^{\frac{1}{3}}\sqrt{(2\pi)}}{(-\frac{1}{3})!} \\ &= \frac{\{e^{k\sigma} k^{\frac{1}{3}} + e^{-k\sigma}(-k)^{\frac{1}{3}}\}\sqrt{\pi}}{\{2M'(y_1)\}^{\frac{1}{3}} 3^{\frac{1}{3}} (-\frac{1}{3})!}, \end{aligned} \quad (41)$$

so that, as $|k| \to \infty$ for real k, by (15) and (36),

$$\Pi(k,y_1) \sim e^{ik(\beta\delta-S)} F(k) M_1^{-1} \beta^{\frac{1}{2}} \frac{2^{\frac{1}{3}} 3^{-\frac{1}{3}}}{\{M'(y_1)\}^{\frac{1}{3}} (-\frac{1}{3})!} |k|^{\frac{1}{3}} e^{\frac{1}{4}\pi i \operatorname{sgn} k}. \quad (42)$$

6. Propagation of a discontinuity

If the incident wave $f(x+\beta y)$ possesses a singularity, such as a discontinuity (as would be needed to represent a shock), this will be associated with a specific asymptotic form of $F(k)$ as $|k| \to \infty$ for real k. This in turn will produce specific asymptotic forms for $G(k)$ and $\Pi(k,y)$, by (38), (39), (40), and (42), which must correspond to singularities of $g(x-\beta y)$ and of $w(x,y)$ which can be determined.

Thus if $f(x)$ includes a discontinuous increase Δ at $x = 0$ (corresponding to a shock of 'strength' $\gamma\Delta$ on the line $x+\beta y = 0$), then $F(k)$ includes a term $-i\Delta/2\pi k$ for real k of large numerical magnitude. It is only asymptotic to this if the discontinuity is the 'worst' singularity for finite x. But assuming this,

$$F(k) \sim -\frac{i\Delta}{2\pi k} \quad (43)$$

as $|k| \to \infty$ for real k.

Hence by (38)
$$G(k) \sim e^{2ik(\beta\delta-S)} \frac{\Delta}{2\pi|k|}. \quad (44)$$

The exponential factor merely indicates that the singularity of $g(x-\beta y)$ is on the line $x-\beta y+2(\beta\delta-S) = 0$, which will be explained physically below. But the factor $1/(2\pi|k|)$ indicates that the character of the singularity has changed, and that

$$g(x-\beta y) = -\frac{\Delta}{\pi} \log[x-\beta y+2(\beta\delta-S)] + \text{continuous function}. \quad (45)$$

Thus, on the linearized theory, the shock is reflected as a positive logarithmic infinity of pressure. Of course this cannot be exactly true, nor does the above constitute a proof that it be true since the conclusion contradicts the hypothesis of small disturbances. But in recent years aerodynamics has provided a large number of examples of a solution of a linearized problem taking infinite values locally. In many of these the quantitative conclusions were found to be sufficiently accurate away from the singularity, while qualitatively the values near the singularity were larger than elsewhere, though not infinite. Thus it may be permissible to conclude that a shock is reflected locally from the boundary layer as a pressure ridge, that is as a rapid compression followed by a rapid expansion. The compression waves might be expected to form a shock-like envelope which would gradually encroach on the fan-like expansion and weaken it.

The reflection of the discontinuity is better understood by observing the singularities of $\varpi(x, y)$ in the supersonic part $y_1 < y \leqslant \delta$ of the boundary layer. By (40) and (43)

$$\Pi(k, y) \sim \frac{M(y)[M^2(y)-1]^{-\frac{1}{4}}}{M_1 \beta^{-\frac{1}{4}}} \left[e^{ik(\beta\delta+s-S)} \left(-\frac{i\Delta}{2\pi k} \right) + e^{ik(\beta\delta-s-S)} \frac{\Delta}{2\pi |k|} \right]. \quad (46)$$

Hence $\varpi(x, y)$ has a discontinuity of magnitude

$$\frac{M(y)[M^2(y)-1]^{-\frac{1}{4}}}{M_1 \beta^{-\frac{1}{4}}} \Delta \quad \text{on} \quad x = -\beta\delta + S - s, \quad (47)$$

while

$$\varpi(x, y) + \frac{\Delta}{\pi} \frac{M(y)[M^2(y)-1]^{-\frac{1}{4}}}{M_1 \beta^{-\frac{1}{4}}} \log(x + \beta\delta - S - s) \quad (48)$$

is continuous at $x = -\beta\delta + S + s$.

The curve along which the singularity is propagated and reflected is shown plain in Fig. 1, where the sonic line and the edge of the boundary layer are shown as broken lines. The shock, after entering the layer at $(-\beta\delta, S)$, follows a characteristic, on which $dx/dy = \{M^2(y)-1\}^{\frac{1}{2}}$ and so, by (30), $x = -\beta\delta + S - s$. It is reflected from its point of incidence on the sonic line $(-\beta\delta + S, y_1)$ along the other characteristic through this point, $x = -\beta\delta + S + s$. Finally it leaves the layer at $(-\beta\delta + 2S, \delta)$, and travels along the characteristic $x - \beta y + 2(\beta\delta - S) = 0$, as was already clear from (45).

But on reflection from the sonic line it changes its character from a pressure 'step' to a pressure 'ridge'. The magnitude of both singularities also increases, as the sonic line is approached, by a factor

$$M(y)[M^2(y)-1]^{-\frac{1}{4}} / M_1 \beta^{-\frac{1}{4}}.$$

This 'focusing' is not very intense. The author attempted a more exact theory of the refraction of a shock entering a region of parallel and otherwise undisturbed steady flow at monotonically decreasing supersonic Mach number, which indicates that such an increase of strength occurs until a state is reached in which the flow behind the shock is subsonic; after this the refraction theory does not apply, but the shock presumably weakens; this phenomenon does not appear on the present theory since no region of subsonic flow behind the shock is admitted.*

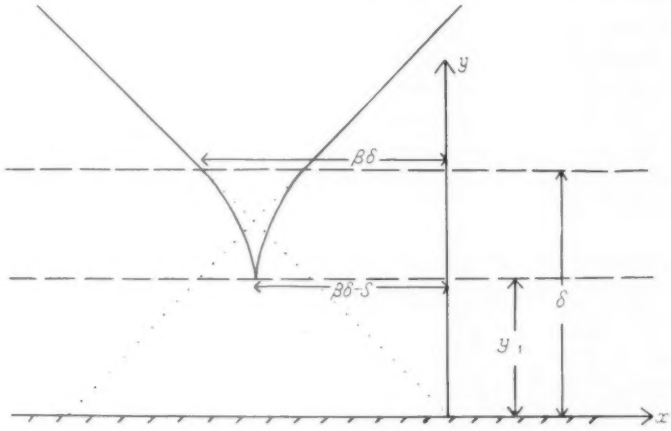

FIG. 1.

The point of incidence on the sonic line $(-\beta\delta+S, y_1)$ is an *algebraic* infinity of pressure in the theory, since both the step and the ridge become infinitely tall, like $[M^2(y)-1]^{-\frac{1}{4}}$, on approaching it. The nature of the infinity, as approached along the sonic line can be deduced from equations (42) and (43). In fact

$$\varpi(x, y_1) \sim \frac{\Delta}{2\pi} M_1^{-1}\beta^{-\frac{1}{2}} \frac{2^{\frac{1}{6}}3^{-\frac{1}{12}+\frac{1}{4}\operatorname{sgn}(x+\beta\delta-S)}}{\{M'(y_1)\}^{\frac{1}{6}}(-\frac{1}{3})!}(-\tfrac{5}{6})!\,|x+\beta\delta-S|^{-\frac{1}{4}} \qquad (49)$$

as x approaches the singularity $-\beta\delta+S$. But in the subsonic layer, as has been seen, the solutions are regular.

To conclude this section, consider an incident wave whose worst singularity is a discontinuity in pressure *gradient*. Then no infinities of pressure arise. The propagation of the singularity is predicted most easily by observing that $\partial\varpi/\partial x$ satisfies exactly the same differential equation

* Though this deficiency of the theory is serious, it is difficult to believe it capable (as hypothesis (iii) of the introduction suggests) of producing the utter disagreement with experiment regarding distance of upstream influence.

and wall boundary condition as ϖ. Hence if, for $y > \delta$, $\partial\varpi/\partial x$ jumps by Δ at the line $x+\beta y = 0$, its resulting singularities in the layer and in the reflected wave will be exactly those given for ϖ itself by equations (45), (47), (48), and (49) above. Those for ϖ may be obtained by integration.

An incident wave consisting of a rapid but not discontinuous rise in pressure could be represented by a jump Δ in $\partial\varpi/\partial x$ followed closely by a jump $-\Delta$ in $\partial\varpi/\partial x$. This would be reflected from the laminar boundary layer (according to the theory) as a local region of high positive $\partial\varpi/\partial x$ followed by a local region of high negative $\partial\varpi/\partial x$—in other words by a ridge. This may be so even if the theory is inadequate when ϖ itself has discontinuities. Again, a Prandtl–Meyer expansion, in which the pressure drops rapidly but not discontinuously, would be reflected as a pressure *trough*, i.e. as an expansion wave followed closely by a compression wave (with the latter probably forming a shock and bordering the expansion region). This prediction is to be tested in the Manchester University Fluid Motion Laboratory.

7. Upstream influence of the disturbance

Assume now that the front of the incident wave is the line $x+\beta y = 0$ shown in Fig. 1. (Then $f(x+\beta y) = 0$ for $x+\beta y < 0$.) It will be investigated how far to the left of this line the disturbance in the boundary layer extends. More precisely the asymptotic behaviour of $\varpi(x, y)$ and $g(x-\beta y)$ as $x \to -\infty$ will be investigated, and the effective distance of upstream influence estimated principally from their logarithmic decrement.

Since $f(x) = 0$ for $x < 0$, $F(k) = (2\pi)^{-1} \int_0^\infty e^{-ikx} f(x)\, dx$ is regular in the lower half-plane. Hence the only singularities of $\Pi(k, y)$ or of $G(k)$ in the lower half-plane, by (15) and (16), are poles at the zeros of their common denominator. Hence whenever $x < 0$ the functions $\varpi(x, y)$ and $g(x)$, by (12), may be expressed as $-2\pi i$ times the sum of the residues of $\Pi(k, y)e^{ikx}$ and $G(k)e^{ikx}$ respectively at the zeros of their common denominator in the lower half-plane.

Since in this section only the lower half of the k-plane will be considered, it is convenient to put $k = -i\kappa$, and

$$D(\kappa) = Q_y(-i\kappa, \delta) + \beta\kappa Q(-i\kappa, \delta) \tag{50}$$

for the denominator of (15) and (16). $D(\kappa)$ is real for real κ since Q involves k only through its square. If the residues of $\Pi(k, y)$ and $G(k)$ at the nth zero κ_n of $D(\kappa)$ in the half-plane are $R_n(y)$ and r_n respectively, then for $x < 0$

$$\varpi(x, y) = -2\pi i \sum_{n=1}^{\infty} R_n(y) e^{\kappa_n x}, \qquad g(x) = -2\pi i \sum_{n=1}^{\infty} r_n e^{\kappa_n x}. \tag{51}$$

The zero κ_1 with least real part gives the logarithmic decrement of ϖ and q as $x \to -\infty$; they decrease by a factor approximately $e^{-\kappa_1}$ per unit distance.

Now as $|\kappa| \to \infty$, by (36) which is valid in the whole of the lower half k-plane,

$$D(\kappa) \sim 4\beta^{\frac{1}{2}} M_1 \{M'(0)\}^{-1} \kappa^{-1} e^{\kappa S} \sin(\kappa\sigma - \tfrac{1}{4}\pi). \tag{52}$$

Hence the zeros are asymptotically at the points $(m+\frac{1}{4})\pi\sigma^{-1}$ for integer m. These zeros for large $|\kappa|$ must be exactly real, since there can only be one in the neighbourhood of each zero of (52), while complex zeros must occur in pairs.

It seems unlikely physically that any zero should be complex, as this would mean that the upstream disturbance was wavy. Also it is more likely that $D(\kappa)$ should be zero for real κ, where $\operatorname{im}\{D(\kappa)\}$ is automatically zero, than for complex κ where $\operatorname{re}\{D(\kappa)\}$ and $\operatorname{im}\{D(\kappa)\}$ must vanish simultaneously. But this point will be returned to later. In the meantime only real zeros of $D(\kappa)$ are considered, and a lower bound for them determined.

It is probable that zeros of $D(\kappa)$ do exist near to the zeros

$$\tfrac{5}{4}\pi\sigma^{-1},\ \tfrac{9}{4}\pi\sigma^{-1},\ \tfrac{13}{4}\pi\sigma^{-1},\ldots,$$

of (52). Since a good estimate of σ is $\frac{1}{4}\pi y_1$ (its value, by (33), when $M(y)$ is a linear function of y in the subsonic layer) these correspond to disturbances with logarithmic decrements of about $5y_1^{-1}$ and more. These would be effectively damped out in a single thickness y_1 of the subsonic layer. Clearly then the only hope of a result in agreement with the experiments is from a zero of $D(\kappa)$ corresponding to the zero $\frac{1}{4}\pi\sigma^{-1}$ of its asymptotic form (51), and preferably a good deal smaller still (say about $\frac{1}{10}\sigma^{-1}$).

It might reasonably be expected that such a zero exists, if it were not that the zero at $\frac{5}{4}\pi\sigma^{-1}$ of the asymptotic form of $D(\kappa)$ is one at which that of $D'(\kappa)$ is

$$D'(\kappa) \sim -4\beta^{\frac{1}{2}} M_1 \{M'(0)\}^{-1} \kappa^{-1} e^{\kappa S} \sigma < 0. \tag{53}$$

This indicates that probably $D'(\kappa) < 0$ at the zero of $D(\kappa)$ near $\kappa = \frac{5}{4}\pi\sigma^{-1}$. On the other hand, $D(0) = Q_y(0, \delta)$. But by the equation and boundary conditions for Q in the simple case $k = 0$, $Q(y) = 3M^2(y)/M'^2(0)$, so that

$$D(0) = 3M_1^2/M'^2(0) > 0. \tag{54}$$

This shows that there can only be an even number of zeros between 0 and the one near $\frac{5}{4}\pi\sigma^{-1}$; and there may well be none.

The possibility of small real zeros of $D(\kappa)$ will now be disproved. In fact it will be shown that if

$$\kappa < \left[\int_0^{y_1} \{M^{-2}(y) - 1\}\, dy \int_0^y M^2(z)\, dz\right]^{-\frac{1}{2}} \tag{55}$$

then $Q(-i\kappa, y)$ is an increasing function for $0 \leqslant y \leqslant \delta$. Hence (50) certainly cannot vanish.

To prove the result, suppose that there *were* values of y at which dQ/dy was not positive, and let y_0 be the least such. Then $Q'(y) > 0$ for $0 < y \leqslant y_0$ but $Q'(y_0) \leqslant 0$.

First assume $y_0 \leqslant y_1$. Then by (55)

$$1-\kappa^2 \int_0^{y_0} \{M^{-2}(y)-1\}\, dy \int_0^{y} M^2(z)\, dz > 0, \tag{56}$$

since the integrand is positive when $y_0 < y < y_1$. Now by equation (13) for Q, and its boundary condition (17),

$$\frac{dQ/dy}{M^2(y)} = \frac{3}{M'^2(0)} - \kappa^2 \int_0^{y} \{M^{-2}(y)-1\} Q(y)\, dy. \tag{57}$$

Hence for $y \leqslant y_0$ (when $Q(y) > 0$, $M(y) \leqslant 1$)

$$\frac{dQ}{dy} \leqslant \frac{3}{M'^2(0)} M^2(y), \qquad Q \leqslant \frac{3}{M'^2(0)} \int_0^{y} M^2(z)\, dz. \tag{58}$$

Hence by (56)

$$1-\tfrac{1}{3}M'^2(0)\kappa^2 \int_0^{y_0} \{M^{-2}(y)-1\} Q(y)\, dy > 0, \tag{59}$$

so that by (57) $Q'(y_0) > 0$, a contradiction.

If alternatively $y_0 > y_1$, then by (57) the inequalities (58) hold when $y \leqslant y_1$ since in this region $Q(y) > 0$ and $M(y) \leqslant 1$. Hence by (55)

$$1-\kappa^2 \int_0^{y_1} \{M^{-2}(y)-1\} \tfrac{1}{3}M'^2(0) Q(y)\, dy > 0. \tag{60}$$

But, since $M(y) > 1$ and $Q(y) > 0$ when $y_1 < y < y_0$, (60) implies (59), and hence again (by (57)) the contradiction $Q'(y_0) > 0$. This completes the proof that no zero of $D(\kappa)$ satisfies (55).

The lower bound (55) for zeros of $D(\kappa)$ always exceeds $y_1^{-1}\sqrt{2}$, since

$$\int_0^{y_1} \{M^{-2}(y)-1\}\, dy \int_0^{y} M^2(z)\, dz < \int_0^{y_1} \left\{ M^{-2}(y) \int_0^{y} M^2(z)\, dz \right\} dy$$

$$< \int_0^{y_1} y\, dy = \tfrac{1}{2}y_1^2. \tag{61}$$

This smallest value $y_1^{-1}\sqrt{2}$ is attained in the limit of a Mach number distribution which is small until just before y_1 and then suddenly shoots up to 1. For any ordinary distribution it would be close to $y_1^{-1}\sqrt{12}$, its value for a linear distribution of Mach number in the subsonic layer ($M(y) = y/y_1$ for $0 \leqslant y \leqslant y_1$). But even a logarithmic decrement $y_1^{-1}\sqrt{2}$ for the upstream disturbance, corresponding to a fourfold decrease per unit thickness y_1 of the subsonic layer, is in complete disagreement with experiment.

Actually it has been shown that under condition (55) Q_y is essentially positive, so that a zero of $D(\kappa)$, by (50), is still remote. This seems to the author overwhelming evidence that the first zero is that near $\frac{5}{4}\pi\sigma^{-1}$, though, as the last paragraph shows, the qualitative result is independent of this conjecture.

However, an argument which supports the conjecture that the only zeros, even complex ones, are at points near

$$\tfrac{5}{4}\pi\sigma^{-1},\ \tfrac{9}{4}\pi\sigma^{-1},\ \tfrac{13}{4}\pi\sigma^{-1},\ldots \tag{62}$$

is based on following the quantity

$$A(k) = \arg[Q_y(k,\delta)+\beta ikQ(k,\delta)] \tag{63}$$

round a large semicircle in the lower half-plane with centre the origin. This quantity is known for large $|k|$ in the lower half-plane by (36), and for $k = 0$ it is zero or some multiple of 2π by (54). Also for small $|k|$ the solution obtained by solving (57) by successive approximations would appear to be fairly smooth. It might be assumed, therefore, that the known value of $A(k)$ at $k = 0$ is joined on to those for large positive and large negative k somewhat smoothly.

Now it may be shown, by equating the change in (63) on following the semicircular path, from its lowest point in the positive sense back to the lowest point, to 2π times the number of encircled zeros of $D(\kappa)$, that the conjecture that there is one zero of $D(\kappa)$ corresponding to each of (62), and that there are no others, is equivalent to a behaviour of $A(k)$ near $k = 0$ as in the broken line in Fig. 2. (Here the plain lines indicate the asymptotes for large $|k|$.) But if an additional zero in the lower half-plane exists, the behaviour would have to be somewhat like that of the dotted line. And if $D(\kappa)$ has complex zeros (which must occur in pairs), a behaviour like that of the dash-dotted line would be necessary. But actually, since $Q_y(0,\delta)$ and $Q(0,\delta)$ are positive, $A(k)$ must increase with k at $k = 0$, by (63). Hence the dash-dotted line must be replaced by a curve with the same asymptotes cutting the axis at the same place but with positive slope. The mathematical unlikelihood of this, combined with the physical

unlikelihood of the upstream disturbance being wavy, convinces the present author that the only zeros are in one-to-one correspondence with those in (62) and approach them more and more closely as $\kappa \to \infty$.

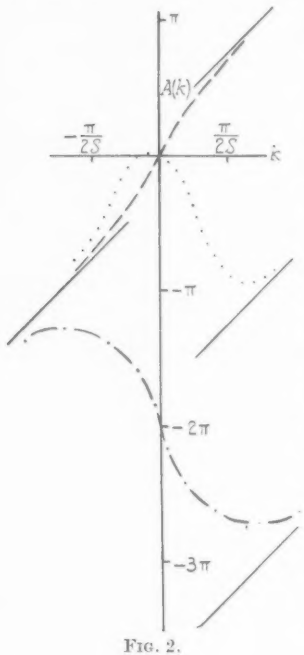

FIG. 2.

Finally the actual term in the series of exponentials for the reflected wave $g(x-\beta y)$, corresponding to the first zero of $D(\kappa)$, is estimated from the asymptotic formulae, in order to quash the possible objection that though the exponential decreases very rapidly its coefficient may be uncommonly vast. Only the reflected wave need be considered, since it is this, not changes in the boundary layer, which is actually observed (by the Schlieren process). It should also be noticed that further terms need not be considered; for no large rate of rise of the coefficients in the series (51) is permissible since each must converge whenever $x < 0$.

When the denominator of $G(k)$ is first zero, very roughly $\kappa = \frac{5}{4}\pi\sigma^{-1}$ and its derivative is given by (53). The numerator of $G(k)$ is not given by (37), which in the lower half-plane neglects a larger and predominant term $O(k^{-2}e^{ikS+k\sigma})$. At the zero this is $O(\kappa^{-2}e^{\kappa S})$. Hence by (16) the residue of $G(k)$ at this zero is only a moderate multiple of

$$e^{5\pi\beta\delta/2\sigma}F\left(-\frac{5i\pi}{4\sigma}\right) = e^{5\pi\beta\delta/2\sigma}\frac{1}{2\pi}\int_0^\infty e^{-5\pi x/4\sigma}f(x)\,dx = e^{5\pi\beta\delta/2\sigma}O(f). \quad (64)$$

Hence, by (51), we have roughly

$$g(x) \sim \exp\left[\frac{5\pi}{4\sigma}(x+2\beta\delta)\right]O(f). \quad (65)$$

Thus reflection along lines upstream of the reflected wave front

$$x-\beta y+2(\beta\delta-S) = 0$$

(which is illustrated in Fig. 1) by more than about $2S+\sigma$ would be negligible. Now $2S+\sigma$ is about two boundary layer thicknesses. In any case the $2S$ portion already corresponds to signals sent back by the incident wave on hitting the outside of the layer.

The conclusion that upstream influence of a weak disturbance on a

'boundary layer of inviscid fluid' is practically negligible, as stated and discussed in the introduction, is therefore inescapable.

8. Examples. Connexion with Tsien and Finston's theory

The author first carried out the work of sections 2–7 for the case of a boundary layer with the Mach number varying linearly across it. For this case $M(y) = y/y_1$ $(0 \leqslant y \leqslant \delta)$, and $\delta = y_1 M_1$, and $Q(k, y)$ is found to be the confluent hypergeometric function

$$Q(k,y) = y^3 e^{-\frac{1}{2}iky^2/y_1} {}_1F_1(\tfrac{5}{4} - \tfrac{1}{4}iky_1; \tfrac{5}{2}; iky^2/y_1). \tag{66}$$

However, he later found that just as much could be discovered in the general case.

More interesting is an attempt to achieve the Tsien and Finston theory (1) as a limiting case of the present theory. In their theory $M(y)$ is a constant $M_2 < 1$ in the whole layer $0 < y < \delta$. This value does not satisfy the conditions of continuity down to $M(0) = 0$ and up to $M(\delta) = M_1$ imposed in this paper (section 2). But it is a limiting case, as $\epsilon \to 0$, of the distribution

$$\left.\begin{aligned} M(y) = M_2 \frac{y}{\epsilon} \quad (0 < y < \epsilon), \qquad & M_2 \quad (\epsilon < y < \delta - \epsilon), \\ & M_1 - (M_1 - M_2)\frac{\delta - y}{\epsilon} \quad (\delta - \epsilon < y < \delta), \end{aligned}\right\} \tag{67}$$

which for $\epsilon > 0$ does satisfy the conditions.

Now in $0 < y < \epsilon$, for this distribution, by (57),

$$\frac{dQ}{dy} = \frac{3M^2(y)}{M'^2(0)}\{1 + O(\epsilon^2)\} = 3y^2\{1 + O(\epsilon^2)\}. \tag{68}$$

Hence at $y = \epsilon$

$$Q(\epsilon) = \epsilon^3 + O(\epsilon^5), \qquad Q'(\epsilon) = 3\epsilon^2 + O(\epsilon^4). \tag{69}$$

But in $\epsilon < y < \delta - \epsilon$, by (13),

$$Q''(y) - k^2\beta_2^2 Q(y) = 0, \tag{70}$$

where $\beta_2 = (1 - M_2^2)^{\frac{1}{2}}$. Hence $Q(y)$ is a combination of $\sinh(k\beta_2 y)$ and $\cosh(k\beta_2 y)$ in this interval, which by (69) must be

$$Q(y) = 3\epsilon^2(k\beta_2)^{-1}\sinh(k\beta_2 y) + O(\epsilon^3). \tag{71}$$

Hence, at $y = \delta - \epsilon$,

$$\left.\begin{aligned} Q(\delta - \epsilon) &= 3\epsilon^2(k\beta_2)^{-1}\sinh(k\beta_2\delta) + O(\epsilon^3), \\ Q'(\delta - \epsilon) &= 3\epsilon^2\cosh(k\beta_2\delta) + O(\epsilon^3). \end{aligned}\right\} \tag{72}$$

But in $\delta-\epsilon < y < \delta$, by (13),

$$\frac{Q'(y)}{M^2(y)} = \frac{Q'(\delta-\epsilon)}{M_2^2} + k^2 \int_{\delta-\epsilon}^{y} \{M^{-2}(y)-1\} Q(y)\, dy = \frac{Q'(\delta-\epsilon)}{M_2^2} + O(\epsilon^3). \quad (73)$$

Hence

$$\left.\begin{aligned} Q'(\delta) &= (M_1^2/M_2^2)Q'(\delta-\epsilon)+O(\epsilon^3) = 3(M_1^2/M_2^2)\epsilon^2 \cosh(k\beta_2\delta)+O(\epsilon^3), \\ Q(\delta) &= Q(\delta-\epsilon)+O(\epsilon^3) = 3\epsilon^2(k\beta_2)^{-1}\sinh(k\beta_2\delta)+O(\epsilon^3). \end{aligned}\right\} \quad (74)$$

Thus, by its definition (50),

$$D(\kappa) = 3(M_1^2/M_2^2)\epsilon^2 \cos(\kappa\beta_2\delta)+\beta\kappa 3\epsilon^2(\kappa\beta_2)^{-1}\sin(\kappa\beta_2\delta)+O(\epsilon^3), \quad (75)$$

and $D(\kappa) = 0$ in the limit as $\epsilon \to 0$ when

$$M_1^2\beta_2 \cos(\kappa\beta_2\delta)+M_2^2\beta\sin(\kappa\beta_2\delta) = 0. \quad (76)$$

Tsien and Finston introduce a parameter θ which in the notation of this paper satisfies

$$\tan\tfrac{1}{2}\theta = \frac{M_1^2\beta_2}{M_2^2\beta}. \quad (77)$$

Then (76) becomes $\sin(\frac{1}{2}\theta+\kappa\beta_2\delta) = 0$, giving for the first zero of $D(\kappa)$

$$\kappa_1 = (\pi-\tfrac{1}{2}\theta)/\delta\beta_2. \quad (78)$$

Since, by (33) in the limit as $\epsilon \to 0$, $\delta\beta_2$ is σ, this root, in lying between $\frac{1}{2}\pi\sigma^{-1}$ and $\pi\sigma^{-1}$, agrees with the general theory of section 7. (The lower bound (55) has become $2^{\frac{1}{2}}\sigma^{-1}$.)

But the logarithmic decrement of the disturbance in Tsien and Finston's theory (**1**) is $(\frac{1}{2}\pi-\frac{1}{2}\theta)/\delta\beta_2$, which can be quite small, by (77), if $M_1^2\beta_2/M_2^2\beta$ is large.

The cause of this is essentially that they take $Q(y)$ proportional to $\cosh(k\beta_2 y)$, not to $\sinh(k\beta_2 y)$, in their layer $0 < y < \delta$ of uniform subsonic Mach number M_2. This is derived from the condition that $v = 0$ at the wall $y = 0$, which (since $U(y) \neq 0$ there) gives that the deflexion $\eta = 0$, and hence by (6) $\partial\varpi/\partial y = 0$.

But as a result of the present limiting process, by (12), (15), (69), and (75), the boundary conditions at the side of the subsonic layer nearest to the wall are of the form $\varpi = 0$, $\partial\varpi/\partial y \neq 0$. Thus $\eta \neq 0$, but the pressure p takes a constant value P. Interpreted physically this means that the very thin vortex layer separating the main subsonic layer from the wall is not of uniform thickness as a result of the disturbance, but is at uniform pressure.

It seems pointless to argue about the relative reasonableness of this condition or of Tsien and Finston's, since the model is in any case artificial. The author's purpose has merely been to show why, though the Tsien and Finston theory is a limiting case of the present theory, the results it gives

on the distance of upstream influence are so very different. The significance of the divergence to the theory of shock boundary layer interaction has already been fully discussed in section 1.

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ON SOME GENERAL THEOREMS IN THE LINEARIZED THEORY OF COMPRESSIBLE FLOW

By F. URSELL *and* G. N. WARD
(*Department of Mathematics, The University, Manchester*)

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SUMMARY

The linearized equations of motion for steady irrotational isentropic flow past a thin body with a sharp trailing edge are used to evaluate the leading terms in the components of the force acting on the body when there is a vortex wake. It is shown that with certain additional assumptions required by the appearance of infinities at the edges of the wake the linearized theory is adequate for the calculation of the lift and drag on a thin body, but not for the calculation of the lateral force, except in special cases, e.g. when the wing is nearly plane.

A quadratic identity involving the drag is used to show that there is at most one solution with given boundary conditions. The corresponding bilinear identity is used to obtain relations between the flow past a thin body and the reversed flow past the same body, when the roles of leading and trailing edges are reversed. These relations hold for both subsonic and supersonic flow and include all flow-reversal theorems obtained by earlier writers.

1. Introduction

GREAT progress has been made in recent years in the theory of the steady two-dimensional flow past thin wings with a sharp trailing edge. For subsonic motions it is found experimentally that the action of viscosity establishes a flow which is smooth near the trailing edge where a thin wake leaves the body. When the flow has become steady, the velocity gradients are not large anywhere, so that ultimately the action of viscosity is small. This situation is idealized by the assumptions that the viscosity vanishes, that the wake is infinitely thin, and that the velocity at the sharp trailing edge is not infinite. These conditions, the last of which is due to Kutta and Joukowski, determine uniquely the flow past the wing. They define a cyclic irrotational motion. The velocity is continuous everywhere, i.e. there is no wake; but the velocity potential is many-valued since there is a circulation about the wing. For supersonic flow there is not, in general, a velocity potential since particles can acquire vorticity by passing through a shock, but when the wing is thin the shock is weak and the vorticity can and will be neglected, so that there is no wake even in this case.

In many cases the incidence, thickness ratio, and camber of a wing are small. A linearized two-dimensional theory has been constructed in which the exact equations and boundary conditions are replaced by linear

approximate equations and boundary conditions which, it is hoped, will give the leading terms in the dynamical variables as incidence, thickness ratio, and camber tend to zero. Where comparison with the exact theory is possible, it has been found that this is the case except near the leading edge where linearized theory gives an infinite velocity. In spite of this discrepancy the linearized theory gives the correct leading term for the forces acting on the wing, for these are given by the momentum integral taken over any curve surrounding the wing, and at a distance from the wing the linearized theory is applicable.

The theory of the flow past a three-dimensional body with a sharp trailing edge is in a less satisfactory state. It is again assumed that viscosity enters only through the condition of Kutta and Joukowski, that the effect of shocks can be neglected, and that the wake leaving the sharp trailing edge is infinitely thin. However, it can no longer be deduced that there is no wake; instead it is assumed that in the ideal flow a vortex sheet leaves the sharp trailing edge. At a distance from the body this sheet is supposed to roll up more and more near the edges until it has ultimately rolled up into two vortex cores of finite diameter. No exact solution of the non-linear equations has yet been found.

When the incidence, thickness ratio, and camber are small it is reasonable to attempt a linearization of the equations. The linearized equations of motion obtained in this way have been solved for linearized boundary conditions in the case of the nearly plane circular wing in incompressible subsonic flow (**1, 2**) and for a wide class of wings in supersonic flow. It is found that infinite velocities occur at the leading edge and at the edges of the vortex sheet.

In the present paper an attempt is made to evaluate the leading terms in the components of the force acting on the body when there is a vortex wake. It will be shown that with certain additional assumptions (required by the appearance of infinities at the edges of the wake) the linearized theory is adequate for the calculation of the lift and drag on a thin body, but only so for the calculation of the lateral force in special cases, e.g. for a nearly plane wing. For these calculations it has been found convenient to derive and discuss afresh the linearized equations of motion and the boundary conditions. The latter cannot be linearized unless the local thickness ratio of the body is small in one direction only. The term 'body' is here used to include a system of isolated bodies.

It is easily seen that the leading term for the drag force is non-negative; this property is used to show that the linearized boundary conditions, including the Kutta–Joukowski condition, make the solution (if any) unique. From the drag formula an integral identity is derived which may

be used to connect a flow past a given body with the reversed flow in which the velocity at infinity is equal in magnitude and opposite in sign so that the role of leading and trailing edges is reversed. Several special cases are known for supersonic flow (**3, 4, 5, 6, 7, 8, 9**). The following simple case is typical: for any thin, nearly plane wing in a supersonic flow, at the trailing edge of which the Kutta–Joukowski condition is satisfied, a reversal of the stream direction leaves the initial slope of the curve of lift against incidence unaltered provided that the Kutta–Joukowski condition is also satisfied at the trailing edge for the new flow. This theorem is here proved under wide conditions and for subsonic flow. Examples can be multiplied indefinitely, a typical more complicated one being that, for any thin body, the rolling moment due to a given rate of yaw is equal to the yawing moment in the reversed flow due to the same rate of roll, the Kutta–Joukowski condition being satisfied at the trailing edges in both flows.

The importance of these theorems lies in the fact that (in supersonic flow at any rate) it is frequently easier to compute the reversed flow rather than the original flow. For instance, to find the lift on a flat delta wing with supersonic leading and trailing edges in supersonic flow, it would be convenient to consider the lift in the reversed flow in which the wing advances with its base forward. Some of the theorems mentioned in this paper have also been given in a recent paper by C. E. Brown (**9**) who generalizes a method due to M. M. Munk (**6**) which involves physical considerations substantially equivalent to the method of the present paper.

The infinite vortex strength at the edges of the wake requires special mention. Its physical meaning is not yet clear. For any finite incidence the strength at the edges in the exact flow must clearly vanish since otherwise there would be finite suction forces acting on the edges. It may be that the maximum vortex strength tends to zero more slowly than the incidence, and that it is located nearer and nearer to the edge, as incidence, thickness ratio, and camber are made to tend to zero at the same rate. The removal of this difficulty requires further study; it would then be possible to justify or refute the additional assumption which is now required for the calculation of the force. The calculation of the lateral force also requires further elucidation. This force is of the second order, and the present paper suggests that the first approximation is inadequate for its calculation except in special cases, e.g. when the body is nearly plane. It is not known how far this apparent inadequacy represents the true state of affairs.

In all this work, much of the detail required for a rigorous mathematical treatment is omitted from considerations of space, but all essential steps

in the arguments are given. For the same reason the final stages in the proofs of the uniqueness theorem are only indicated and appropriate references are given. Vector methods are used throughout.

2. The equation of motion

Let p, ρ, and $\mathbf{u}$ be respectively the pressure, density, and particle velocity at a point in a steady flow of an inviscid gas, then the equations of motion, expressing the conservation of mass and momentum, are

$$\nabla.(\rho\mathbf{u}) = 0 \tag{1}$$

$$\mathbf{u}.\nabla\mathbf{u}+(1/\rho)\nabla p = 0. \tag{2}$$

These equations are too general for the present purpose, and it is assumed that the motion is irrotational and isentropic. The condition of irrotationality is expressed by

$$\nabla\wedge\mathbf{u} = 0, \tag{3}$$

so that $\mathbf{u}$ is the gradient of a scalar point function, the velocity potential. Since the flow is isentropic there is a functional relationship between the pressure and the density, and with the aid of (3), (2) can be integrated to give Bernoulli's equation

$$\int dp/\rho+\tfrac{1}{2}\mathbf{u}^2 = \text{constant}, \tag{4}$$

which is an expression of the conservation of energy. Also the local acoustic velocity, a, at any point is given by

$$a^2 = dp/d\rho. \tag{5}$$

The velocity $\mathbf{u}$ at any point is now expressed in the form

$$\mathbf{u} = \mathbf{U}+\mathbf{v}, \tag{6}$$

where $\mathbf{U}$ is the velocity a long way ahead of the body, and it is assumed that squares and higher powers and products of v/U and its derivatives can be neglected in the equations of motion. Thus the departure of the stream from uniform constant conditions is expressed as a small perturbation, and the linearized equations for steady irrotational flow of a compressible fluid are obtained.

Let p_0, ρ_0, and $\mathbf{U}$ be respectively the pressure, density, and velocity of the undisturbed stream, then, for sufficiently small departures from uniform flow,

$$\int_{p_0}^{p}\frac{dp}{\rho} = \int_{p_0}^{p}\left\{\frac{1}{\rho_0}+(p-p_0)\left[\frac{d}{dp}\left(\frac{1}{\rho}\right)\right]_0+\ldots\right\}dp$$

$$= \frac{p-p_0}{\rho_0}-\frac{1}{2}\left(\frac{p-p_0}{\rho_0}\right)^2\left(\frac{dp}{d\rho}\right)_0+\ldots. \tag{7}$$

From (4), (5), and (7),

$$\frac{p-p_0}{\rho_0}-\frac{1}{2a_0^2}\left(\frac{p-p_0}{\rho_0}\right)^2+\ldots=\tfrac{1}{2}\mathbf{U}^2-\tfrac{1}{2}(\mathbf{U}+\mathbf{v})^2$$
$$=-\mathbf{U}.\mathbf{v}-\tfrac{1}{2}\mathbf{v}^2, \tag{8}$$

and therefore

$$\frac{p-p_0}{\rho_0}=-\mathbf{U}.\mathbf{v}-\tfrac{1}{2}\mathbf{v}^2+\frac{1}{2a_0^2}(\mathbf{U}.\mathbf{v})^2+\ldots$$
$$=-\mathbf{U}.\mathbf{v}-\tfrac{1}{2}\mathbf{v}.\mathbf{w}+\ldots, \tag{9}$$

where
$$\mathbf{w}=\mathbf{v}-\mathbf{U}\mathbf{U}.\mathbf{v}/a_0^2\equiv\boldsymbol{\Phi}.\mathbf{v}, \text{ say}, \tag{10}$$

$\boldsymbol{\Phi}$ being the self-conjugate dyadic (cf. **10**) $\mathbf{I}-\mathbf{U}\mathbf{U}/a_0^2$. The vector $\mathbf{w}$ is proportional to the perturbation mass flux vector, as is shown later. The density ρ is expanded:

$$\rho=\rho_0+(p-p_0)(d\rho/dp)_0+\ldots. \tag{11}$$

which by virtue of (9) becomes

$$\rho=\rho_0(1-\mathbf{U}.\mathbf{v}/a_0^2+\ldots), \tag{12}$$

and hence

$$\rho\mathbf{u}=\rho_0(1-\mathbf{U}.\mathbf{v}/a_0^2+\ldots)(\mathbf{U}+\mathbf{v})$$
$$=\rho_0(\mathbf{U}+\mathbf{v}-\mathbf{U}\mathbf{U}.\mathbf{v}/a_0^2+\ldots)$$
$$=\rho_0(\mathbf{U}+\mathbf{w})+\ldots. \tag{13}$$

Thus, on linearization, equation (1) becomes

$$\nabla.\mathbf{w}=0. \tag{14}$$

Equation (13) gives the physical interpretation of $\mathbf{w}$, since $\rho\mathbf{u}$ is the mass flux vector and, to the first order,

$$\mathbf{U}+\mathbf{w}=(\rho/\rho_0)\mathbf{u}. \tag{15}$$

The vector $\mathbf{U}+\mathbf{w}$ is the mass flux vector divided by the undisturbed stream density, so that $\mathbf{w}$ is the perturbation flux vector. The introduction of the vector $\mathbf{w}$ is due to A. Robinson (**11**), who called it the reduced current velocity. Thus the linearized equations for steady irrotational isentropic flow of a compressible inviscid fluid are

$$\mathbf{u}=\mathbf{U}+\mathbf{v}, \tag{16}$$
$$\nabla\wedge\mathbf{v}=0, \tag{17}$$
$$\nabla.\mathbf{w}=0, \tag{18}$$
$$\mathbf{w}=\boldsymbol{\Phi}.\mathbf{v}=\mathbf{v}-\mathbf{U}\mathbf{U}.\mathbf{v}/a_0^2. \tag{19}$$

Equations (16) to (19) are independent of any particular coordinate frame of reference, but, by introducing a right-handed system of orthogonal Cartesian coordinates x, y, z with x-axis in the direction of the undisturbed stream, and a potential ϕ defined by

$$\mathbf{v}=\nabla\phi, \tag{20}$$

they can be written in the form

$$\mathbf{u} = (U+\phi_x, \phi_y, \phi_z), \tag{21}$$

$$\mathbf{w} = \{(1-M^2)\phi_x, \phi_y, \phi_z\}, \tag{22}$$

$$(1-M^2)\phi_{xx}+\phi_{yy}+\phi_{zz} = 0, \tag{23}$$

where $M = U/a_0$ is the Mach number of the undisturbed stream, and suffixes denote partial derivatives. If the motion be subsonic ($M < 1$), (23) or (17) to (19) are elliptic, and if the motion be supersonic ($M > 1$), the equations are hyperbolic and real characteristic surfaces exist.

The self-conjugate dyadic $\mathbf{\Phi}$ can be expressed in matrix form for this coordinate system as

$$\begin{bmatrix} 1-M^2 & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}.$$

3. The boundary conditions

In formulating boundary conditions for linearized theory it is necessary to distinguish between bodies whose local thickness in any direction normal to the undisturbed stream is small (compared with the length of the body), and bodies whose local thickness normal to the undisturbed stream is small in one direction only at any given point. Bodies of the first kind are here called slender bodies and bodies of the second kind are called thin bodies. For example a pointed body of revolution is a slender body, while nearly plane wings and quasi-cylindrical bodies are thin bodies. There are some bodies, such as a pointed body of revolution with attached thin wings, which are slender in some part (the body) and thin in another (the wings). In what follows only thin bodies of finite overall dimensions are considered, and the following boundary conditions apply only to such bodies.

By definition the boundaries of thin bodies lie at a small distance from some surface whose generators are parallel to **U**, and the projection of the actual body surface on to this surface is called the mean body surface. The boundary conditions on the body surface are applied on this mean body surface. If $\mathbf{n}^*$ be the unit normal to the body surface and $\mathbf{n}$ be the unit normal to the mean body surface, both being inward normals; or, for other surfaces, if $\mathbf{n}$ be the appropriate normal, then the boundary conditions to be satisfied by the perturbation velocity are

(i) on the mean body surface

$$\mathbf{n}.\mathbf{v} = -\mathbf{n}^*.\mathbf{U} \qquad (\mathbf{n}.\mathbf{U} = 0),$$

(ii) on a mean surface corresponding to a free boundary outside which the gas is at rest with given pressure,

$$\mathbf{U}.\mathbf{v} = \text{known constant} \qquad (\mathbf{n}.\mathbf{U} = 0),$$

(iii) at sharp trailing edges of the body $\mathbf{v}$ is finite,

(iv) in the wake of the body, any vortex sheets extending downstream from the sharp trailing edges are surfaces whose generators are parallel to $\mathbf{U}$, with edges also parallel to $\mathbf{U}$, at which $\mathbf{n}.\mathbf{v}$ and $\mathbf{U}.\mathbf{v}$ are continuous,

(v) in supersonic flow, on characteristic surfaces at which $\mathbf{v}$ is discontinuous, $\mathbf{n} \wedge \mathbf{v}$ and $\mathbf{n}.\mathbf{v}$ are continuous,

(vi) at all other points in the fluid, $\mathbf{v}$ is continuous,

(vii) in subsonic flow, if r be distance from the nearest point of the body or the vortex sheet (if any), $r^2 v$ is bounded as $r \to \infty$, and, in supersonic flow, $\mathbf{v} = 0$ on any plane normal to $\mathbf{U}$ and upstream from the body,

(viii) in subsonic flow, if R be distance from the nearest point of the body, $R^2 \mathbf{U}.\mathbf{v}$ is bounded as $R \to \infty$,

(ix) the over-all aerodynamic force on the body is finite (see section 4).

These boundary conditions are not all independent, but it is convenient to have a complete list for later reference.

The justification of the conditions (i) to (ix) is as follows.

(i) Since the fluid under consideration is inviscid, it can slip against the boundaries, and the boundary condition at the body surface is

$$\mathbf{n}^*.(\mathbf{U}+\mathbf{v}) = 0, \tag{24}$$

which is the mathematical expression of the assumption that the fluid does not separate from the body. Since the body is thin and the body surface everywhere makes a small angle with the direction of the undisturbed stream, $\mathbf{n}^*.\mathbf{U}$ must be small compared with U, and the difference between $\mathbf{n}^*.\mathbf{v}$ on the body and $\mathbf{n}.\mathbf{v}$ on the mean surface is a small quantity of the second order which may be neglected to give the linearized boundary condition (i) from (24). The solution of the linearized equations of motion with this boundary condition can lead to infinite values of v at sharp edges, and the condition $v \ll U$ is clearly violated. This difficulty is discussed in section 4. This complete linearization of the boundary condition at the body surface is not possible for slender bodies.

(ii) On free boundaries where the gas is at rest on one side, the condition of constant pressure may be expressed by the linearized approximation to Bernoulli's equation in the form

$$\mathbf{U}.\mathbf{v} = \text{known constant}, \tag{25}$$

and, if the free boundary extend upstream to infinity, the constant is zero. This boundary condition can be applied on the mean surface of the free boundary whose generators are parallel to $\mathbf{U}$, to give a completely linearized boundary condition as before without causing loss of accuracy.

(iii) At sharp trailing edges it is necessary to apply the condition that v must be bounded there in accordance with the Kutta–Joukowski hypothesis in order that the mathematical solution shall agree with physical experience. It is found that with this condition the velocity at sharp leading edges becomes infinite in general, as has been mentioned under (i) above.

(iv) A result of applying this Kutta–Joukowski condition is that surfaces on which a tangential component of velocity is discontinuous appear in the mathematical solution. Such surfaces of discontinuity, or vortex sheets, are not incompatible with the assumption that the fluid is inviscid, but certain physical conditions must be satisfied on them. These conditions are that the velocity must be purely tangential, and that the pressure must be continuous through the vortex sheet. If $\mathbf{v}_1$, $\mathbf{v}_2$ be the perturbation velocities on either side of the sheet, and $\mathbf{n}$ be the normal to the sheet, then these conditions are

$$\mathbf{n}.(\mathbf{U}+\mathbf{v}_1) = \mathbf{n}.(\mathbf{U}+\mathbf{v}_2) = 0, \tag{26}$$

and, if the perturbation velocities be small compared with U,

$$\mathbf{U}.\mathbf{v}_1+\tfrac{1}{2}\mathbf{v}_1.\mathbf{w}_1 = \mathbf{U}.\mathbf{v}_2+\tfrac{1}{2}\mathbf{v}_2.\mathbf{w}_2 \tag{27}$$

from (10). Equation (26) shows that $\mathbf{U}.\mathbf{n}$ must be small compared with U, so that $\mathbf{n}$ is approximately perpendicular to $\mathbf{U}$. Equation (27) may be written

$$(\mathbf{v}_1-\mathbf{v}_2).\{\mathbf{U}+\tfrac{1}{2}(\mathbf{w}_1+\mathbf{w}_2)\} = 0, \tag{28}$$

or, if $\boldsymbol{\omega}$ be the surface vorticity,

$$\mathbf{n}\wedge\boldsymbol{\omega}.\{\mathbf{U}+\tfrac{1}{2}(\mathbf{w}_1+\mathbf{w}_2)\} = 0. \tag{29}$$

Thus, either $\boldsymbol{\omega} = 0$ and there is no vortex sheet at all, or $\mathbf{n}$, $\boldsymbol{\omega}$, and $\mathbf{U}$ are approximately coplanar. Since $\mathbf{n}$ and $\mathbf{w}$ are orthogonal, and $\mathbf{n}$ and $\mathbf{U}$ are approximately so, $\boldsymbol{\omega}$ must be approximately parallel to $\mathbf{U}$. It is usual to adopt Prandtl's assumption that the vorticity $\boldsymbol{\omega}$ is exactly parallel to $\mathbf{U}$ at all points on the vortex sheet, so that the vortex sheet is a surface with generators and edges parallel to $\mathbf{U}$ and which extends from the trailing edge causing it to go to infinity downstream. On this assumption, only the linear approximation to (27), namely

$$\mathbf{U}.\mathbf{v}_1 = \mathbf{U}.\mathbf{v}_2, \tag{27'}$$

is satisfied on this surface in general.

The boundary conditions on the new surface, the linearized vortex sheet, are

$$\mathbf{v}_1.\mathbf{n} = \mathbf{v}_2.\mathbf{n}, \tag{26'}$$

equation (27′), and $\boldsymbol{\omega}$ = constant on lines parallel to $\mathbf{U}$.

It is found that v becomes infinite in general at any edges of the vortex sheet on these assumptions, so that the condition (27′) is not valid in this form near to and at the edges, and the above assumptions become difficult to justify. In fact, the wake cannot extend to infinity as a vortex sheet as has been assumed, since (26) and (26′) show that the streamlines do not lie in the assumed vortex sheet even on linearized theory. In more exact theory it is assumed that a steady motion exists in which the vortex sheet rolls up behind the body to form vortex cores. It seems reasonable to assume further that the rate of rolling up depends upon the vortex strength: at a fixed distance behind the body the vortex wake will ultimately have the linearized form as the strength tends to zero, while on the other hand, for a given strength, a distance can always be found at which the vortex sheet has rolled up to a given extent. On the present assumptions the linearized theory represents an approximation valid in a region which increases as the strength decreases, but not valid uniformly in the whole space.

(v) In supersonic flow real characteristic surfaces exist on which $\mathbf{v}$ may be discontinuous. Consideration of the conservation of mass and momentum through such a surface leads to the conclusions that the normal component of $\mathbf{w}$ and the tangential component of $\mathbf{v}$ must be continuous there. Conservation of energy shows that the constant in Bernoulli's equation (4) is continuous so that the expansion (10) is valid on both sides of such a surface.

(vi) Discontinuities of $\mathbf{v}$ can occur only on vortex sheets and real characteristic surfaces, both of which have been considered.

(vii) From Kelvin's circulation theorem, the total vorticity in all the vortex sheets must be zero so that, at large distances from the body and the vortex wake, the flow must approximate to that of a semi-infinite vortex pair. The given conditions for subsonic flow follow from this if the vortex pair be of finite strength.

In supersonic flow, disturbances cannot be propagated upstream from the body so that the flow there must be that of the undisturbed stream and $\mathbf{v} = 0$. In fact, since $\mathbf{v}$ and its derivatives can be discontinuous only on characteristic surfaces, $\mathbf{v} = 0$ everywhere upstream from the envelope of downstream characteristic semi-cones with vertices on the leading edges of the body.

(viii) This condition follows from the condition (iv) since the vortex lines are of constant strength and parallel to $\mathbf{U}$.

(ix) This condition is clearly necessary if the solution is to be physically significant. Its consequences are discussed in section 4. For subsonic flow it is implicitly contained in conditions (vii) and (viii).

4. The aerodynamic forces

A difficulty arises in the calculation of the aerodynamic forces on the linearized theory at points where the perturbation velocity becomes infinite, since the usual approximation (either linear or quadratic) to Bernoulli's equation is certainly not valid at such points, and the appropriate form for the pressure has to be determined from other considerations.

For steady inviscid flow with no impressed force field, Euler's momentum theorem for any closed surface inside which $\mathbf{u}$ is not singular is

$$\int (p\mathbf{n}+\rho\mathbf{u}\mathbf{u}.\mathbf{n})\,dS = 0. \tag{30}$$

Here the closed surface is taken to consist of the body surface Σ^*, a surface S completely enclosing the body, and a surface σ^* which lies close to both sides of any vortex sheet in the flow between Σ^* and S. If the velocity is finite on the body, the contribution from Σ^* is

$$\mathbf{F} = \int_{\Sigma^*} p\mathbf{n}\,dS \tag{31}$$

since $\mathbf{u}.\mathbf{n}^* = 0$ on the body surface; thus $\mathbf{F}$ is the aerodynamic force on the body. On the surface σ^*, since $\mathbf{n}.\mathbf{u}$ and p are continuous through a vortex sheet, the integral vanishes when taken over both sides of the sheet. Hence

$$\mathbf{F} = -\int_S (p\mathbf{n}+\rho\mathbf{u}\mathbf{u}.\mathbf{n})\,dS, \tag{32}$$

$\mathbf{n}$ being the outward unit normal to S. This integral is invariant for changes of S. When there are sharp edges on the body, at which the velocity becomes infinite, there are finite suction forces on such edges. Continuity considerations indicate that the total force $\mathbf{F}$ is to be defined by (32) rather than by (31). In this paper it will therefore be supposed that the aerodynamic force is given by (32).

If a surface S exists on which the perturbation velocity is uniformly small compared with U, as is assumed in references (**3**) and (**12**), the integrand in (32) may be approximated by using the expansion (9) for the pressure, the expansion (12) for the density, and neglecting third-order terms. Then, on taking account of mass conservation inside S, (32) becomes approximately

$$\begin{aligned}\mathbf{F} &= \rho_0 \int_S \{(\mathbf{U}.\mathbf{v}+\tfrac{1}{2}\mathbf{v}.\mathbf{w})\mathbf{n}-[(1-\mathbf{U}.\mathbf{v}/a_0^2)\mathbf{U}.\mathbf{n}+\mathbf{v}.\mathbf{n}]\mathbf{v}\}\,dS \\ &= \rho_0 \int_S \{(\mathbf{U}.\mathbf{v}+\tfrac{1}{2}\mathbf{v}.\mathbf{w})\mathbf{n}-(\mathbf{U}+\mathbf{w}).\mathbf{n}\mathbf{v}\}\,dS. \end{aligned} \tag{33}$$

If $\mathbf{v}$ and $\mathbf{w}$ satisfy the linearized equations of flow and the boundary conditions of section 3, this is the momentum integral for linearized theory

as given in (**4**) and (**11**), and it is shown there that this integral is invariant for changes of S such that S always encloses the same singularities, since the divergence of the integrand vanishes in linearized theory. However, in a flow with a vortex wake satisfying the boundary condition (iv) of section 3, the integral in (33) is not invariant for all changes of S such that S encloses the body, as is assumed in (**3**) and (**12**), since integrals of the form (33) taken over two surfaces S and S' differ by the value of a similar integral taken over that part of the linearized vortex sheet which lies between S and S'. This last integral does not vanish in general, because the velocity at the edges of the vortex sheet is infinite on the linearized theory, and there is a contribution which depends upon the quadratic terms in (33). Thus the integral (33) is not a valid approximation to the integral (32), and does not give the forces correctly in linearized theory. These difficulties which arise in the application of the momentum method to the calculation of the aerodynamic forces are mostly due to the assumption of the idealized form taken for the vortex sheet in linearized theory, and it is necessary to consider the approximation to the integral (32) in more detail.

Let $\mathbf{v}^*$ and $\mathbf{w}^*$ be the exact perturbation velocity and the exact perturbation flux vector respectively for flow past a thin body, so that

$$\mathbf{u} = \mathbf{U}+\mathbf{v}^*, \qquad \rho\mathbf{u} = \rho_0(\mathbf{U}+\mathbf{w}^*). \tag{34}$$

Then (32) can be written

$$\mathbf{F} = \int_S \{[\rho_0\,\mathbf{U}.\mathbf{v}^*-(p-p_0+\rho_0\mathbf{U}.\mathbf{v}^*)]\mathbf{n}-\rho_0(\mathbf{U}+\mathbf{w}^*).\mathbf{n}\mathbf{v}^*\}\,dS. \tag{35}$$

If ϕ^* is the exact perturbation potential for subsonic flow or is the perturbation potential which gives $\mathbf{v}^*$ to the second order in supersonic flow, then, either exactly in the former case, or to the second order in the latter case,

$$\int_S (\mathbf{U}.\mathbf{v}^*\mathbf{n}-\mathbf{U}.\mathbf{n}\mathbf{v}^*)\,dS = \int_S \mathbf{U}\wedge(\mathbf{n}\wedge\mathbf{v}^*)\,dS$$
$$= \mathbf{U}\wedge\int_S \mathbf{n}\wedge\nabla\phi^*\,dS = \mathbf{U}\wedge\int_{\gamma^*}\phi^*\,d\mathbf{s}, \tag{36}$$

where the line contour γ^* is both sides of the curve in which the vortex wake intersects S for the exact flow. Equation (35) then becomes

$$\mathbf{F} = \rho_0\,\mathbf{U}\wedge\int_{\gamma^*}\phi^*\,d\mathbf{s}-\int_S\{(p-p_0+\rho_0\,\mathbf{U}.\mathbf{v}^*)\mathbf{n}+\rho_0\,\mathbf{v}^*\mathbf{w}^*.\mathbf{n}\}\,dS. \tag{37}$$

It is not difficult to show that (33) can be written in a similar form, namely,

$$\mathbf{F} = \rho_0\,\mathbf{U}\wedge\int_\gamma\phi\,d\mathbf{s}-\rho_0\int_S(\mathbf{v}\mathbf{w}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{w}\mathbf{n})\,dS, \tag{38}$$

where ϕ is the linearized perturbation potential and γ is both sides of the curve in which the linearized vortex sheet intersects S.

The linearized theory tries to represent the limiting flow when the thickness, camber, twist, incidence, etc., of the body all tend to zero at proportional rates, but the approach to the limit is not uniform all over the fluid, as has been pointed out already in the discussion of the vortex wake in section 3. The perturbation velocities become small compared with U except at leading subsonic edges which become sharp in the limit, and at the edges of the vortex sheet. The perturbation potential becomes small uniformly over the whole field in the limit. At points on S which are not close to the edges of the vortex wake, the integrand of the surface integral in (37) tends to the value of the integrand of the corresponding surface integral in (38), as is seen by using the expansion (9) for the pressure: that is

$$\rho_0 \mathbf{v}^*\mathbf{w}^*.\mathbf{n}+(p-p_0+\rho_0 \mathbf{U}.\mathbf{v}^*)\mathbf{n} \to \rho_0(\mathbf{vw}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{wn}), \tag{39}$$

but this is not so in neighbourhoods of the edges of the vortex wake. If it can be assumed that, in the limit, the contributions of the two expressions in (39) are equal (and it is not known whether this assumption is justified), then

$$\int_S \{\rho_0 \mathbf{v}^*\mathbf{w}^*.\mathbf{n}+(p-p_0+\rho_0 \mathbf{U}.\mathbf{v}^*)\mathbf{n}\}\, dS \to \rho_0 \int (\mathbf{vw}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{wn})\, dS, \tag{40}$$

the difference between the two integrals being of the third order.

The difference between the two line integrals in (37) and (35) is of the second order (since the difference between the line elements on γ^* and γ is of the first order), and this difference is thus of the same order as the surface integrals.

In general the integral over γ^* is difficult to evaluate except when $\gamma^* = \Gamma^*$, where Γ^* is both sides of the trailing edge of the body. The surface S must then be a surface S^* which passes through the trailing edge of the body. Finally, then, the force on the body to the second order is

$$\mathbf{F} = \rho_0 \mathbf{U} \wedge \int_{\Gamma^*} \phi\, d\mathbf{s} - \rho_0 \int_{S^*} (\mathbf{vw}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{wn})\, dS + \rho_0 \mathbf{U} \wedge \int_{\Gamma^*} (\phi^*-\phi)\, d\mathbf{s}, \tag{41}$$

or

$$\mathbf{F} = \rho_0 \int_{S^*} \{(\mathbf{U}.\mathbf{v}+\tfrac{1}{2}\mathbf{v}.\mathbf{w})\mathbf{n}-(\mathbf{U}+\mathbf{w}).\mathbf{nv}\}\, dS + \rho_0 \mathbf{U} \wedge \int_{\Gamma^*} (\phi^*-\phi)\, d\mathbf{s}. \tag{42}$$

The surface integral in (42) is that in (33) for the special case when $S = S^*$. Since the divergences of the integrands of the surface integrals in (41) and (42) vanish, the surface S^* can be transformed into any surface which encloses the body and passes through its trailing edge: in particular S^* can be the actual body surface Σ^* in (41) and (42), or the mean body

surface Σ plus a surface enclosing the sharp leading edge (where the assumptions of linearized theory are certainly not valid) in (41). Thus, on neglecting third-order quantities,

$$\mathbf{F} = -\rho_0 \int_{\Sigma^*} (\mathbf{U}.\mathbf{v}+\tfrac{1}{2}\mathbf{v}.\mathbf{w})\mathbf{n}^*\, dS+\rho_0\,\mathbf{U}\wedge\int_{\Gamma^*}(\phi^*-\phi)\, d\mathbf{s}$$

$$= \int_{\Sigma^*} (p-p_0)\mathbf{n}^*\, dS+\rho_0\,\mathbf{U}\wedge\int_{\Gamma^*}(\phi^*-\phi)\, d\mathbf{s}, \tag{43}$$

where $(p-p_0)$ is the pressure given by the quadratic approximation (10) to Bernoulli's equation, or

$$\mathbf{F} = \rho_0\,\mathbf{U}\wedge\int_{\Gamma^*}\phi\, d\mathbf{s}-\rho_0\int_{\Sigma}(\mathbf{vw}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{wn})\, dS+\rho_0\,\mathbf{U}\wedge\int_{\Gamma^*}(\phi^*-\phi)\, d\mathbf{s}. \tag{44}$$

In general the calculation of the force correct to the second order requires a knowledge of the discontinuities of ϕ^* correct to the second order. In equations (41) to (44) for the force, the term involving ϕ^* has been separated so that it occurs as a second-order term at most, the first-order forces being given by the other terms in these equations. For some special bodies, the component of the term involving $\phi^*-\phi$ which is perpendicular to the first-order force can be shown to be of the third order, so that the lateral force is obtained correctly to the second order from the linearized solution alone. Such a case is that of a nearly plane wing, for which the component of $d\mathbf{s}$ on Γ^* normal to the mean plane is of the first order, so that the lateral force parallel to the mean plane is given correctly to the second order by linearized theory.

In all cases the drag force parallel to $\mathbf{U}$ does not involve the line integrals and is always a second-order force which is given from the linearized solution alone.

When evaluating the second-order forces, care must be taken to include the force at any sharp edges where the velocity is infinite. If r be distance measured normally from a sharp leading edge where the velocity is infinite, then the boundary condition (ix), which requires the aerodynamic force to be finite, limits the velocity at almost all points of the edge to be $\sim Kr^{-\frac{1}{2}}+o(r^{-\frac{1}{2}})$ as $r\to 0$ where K is a finite quantity independent of r. If ψ be the angle made by the tangent to the edge with the undisturbed steam, then it can be shown (cf. Appendix to ref. **13**) that the contribution to $\mathbf{F}$ per unit length of edge is

$$\pi\rho_0 K^2/(1-M^2\sin^2\psi)^{\frac{1}{2}} \tag{45}$$

directed normally outwards from the edge. This force occurs only on leading subsonic edges, where $M^2\sin^2\psi < 1$.

A similar analysis can be carried out for the moment **T** of the aerodynamic pressures about an origin of radii vectores **R**, and leads to the results:

$$\mathbf{T} = \int_{\Sigma^*} (p-p_0)\mathbf{R}\wedge\mathbf{n}^*\,dS+\rho_0\int_{\Gamma^*}(\phi^*-\phi)\mathbf{R}\wedge(\mathbf{U}\wedge d\mathbf{s}), \tag{46}$$

or

$$\mathbf{T} = \rho_0\int_{S^*}\{(\mathbf{U}.\mathbf{v}+\tfrac{1}{2}\mathbf{v}.\mathbf{w})\mathbf{R}\wedge\mathbf{n}-(\mathbf{U}+\mathbf{w}).\mathbf{n}\mathbf{R}\wedge\mathbf{v}\}\,dS+ \\ +\rho_0\int_{\Gamma^*}(\phi^*-\phi)\mathbf{R}\wedge(\mathbf{U}\wedge d\mathbf{s}), \tag{47}$$

or

$$\mathbf{T} = \rho_0\int_{\Gamma^*}\phi\mathbf{R}\wedge(\mathbf{U}\wedge d\mathbf{s})-\rho_0\int_{\Sigma}(\mathbf{R}\wedge\mathbf{v}\mathbf{w}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{w}\mathbf{R}\wedge\mathbf{n})\,dS+ \\ +\rho_0\int_{\Gamma^*}(\phi^*-\phi)\mathbf{R}\wedge(\mathbf{U}\wedge d\mathbf{s}), \tag{48}$$

$(p-p_0)$ in (46) being the quadratic approximation for the pressure.

It is stressed again that the identification of these expressions (41) to (48) with the aerodynamic force and moment requires the assumption about the contributions from the flow near the edges of the vortex sheet to the integral over S^* which was made earlier in this section.

For later references the following results are required. The first-order terms in the expressions (44) and (48) for the force and moment are respectively

$$\mathbf{F}_1 = \rho_0\mathbf{U}\wedge\int_\Gamma\phi\,d\mathbf{s} = -\rho_0\int_\Sigma\mathbf{U}.\mathbf{v}\mathbf{n}\,dS = \int_\Sigma(p_1-p_0)\mathbf{n}\,dS, \tag{49}$$

and

$$\mathbf{T}_1 = \rho_0\int_\Gamma\phi\mathbf{R}\wedge(\mathbf{U}\wedge d\mathbf{s}) = -\rho_0\int_\Sigma\mathbf{U}.\mathbf{v}\mathbf{R}\wedge\mathbf{n}\,dS = \int_\Sigma(p_1-p_0)\mathbf{R}\wedge\mathbf{n}\,dS, \tag{50}$$

where Γ is the trailing edge of the mean body surface Σ, and (p_1-p_0) is the pressure given by the linear approximation to Bernoulli's equation. These are easily derived from (44) and (48), the difference between the integrals over Γ^* and Γ being of the second order. That part of the drag force (in the direction of **U**) which neglects the leading edge suction force, D_1 say, is given by

$$UD_1 = \rho_0\int_\Sigma\mathbf{U}.\mathbf{v}\mathbf{w}.\mathbf{n}\,dS = -\rho_0\int_\Sigma\mathbf{U}.\mathbf{v}\mathbf{U}.\mathbf{n}^*\,dS \\ = \mathbf{U}.\int_\Sigma(p_1-p_0)\mathbf{n}^*\,dS \tag{51}$$

since $\mathbf{w}.\mathbf{n} = \mathbf{v}.\mathbf{n} = -\mathbf{U}.\mathbf{n}^*$ on Σ, and where (p_1-p_0) is the pressure given by the linear approximation to Bernoulli's equation as before.

5. Some integral identities for linearized compressible flow

Consider the quadratic terms in the momentum integral:

$$\int (\mathbf{vw}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{wn})\,dS. \tag{52}$$

If $\mathbf{v}$ and $\mathbf{w}$ satisfy the linearized equations (17), (18), and (19) and are bounded and continuous on, and inside, a closed surface S, then it can be shown that this integral (52) vanishes over S, as has been mentioned before, in section 4. It will now be shown that this result is still true if, in supersonic flow, $\mathbf{v}$ be discontinuous at a characteristic surface of the linearized equations. The integrand in (52) may be written

$$\tfrac{1}{2}\mathbf{vw}.\mathbf{n}+\tfrac{1}{2}\mathbf{w}\wedge(\mathbf{v}\wedge\mathbf{n}), \tag{53}$$

and if the discontinuity in $\mathbf{v}$ is $\Delta v\mathbf{n}$ ($\mathbf{n}\wedge\mathbf{v}$ is continuous), then the discontinuity in this expression (53) is

$$\begin{aligned}\tfrac{1}{2}\Delta v\mathbf{nw}.\mathbf{n}+\tfrac{1}{2}\Delta\mathbf{w}\wedge(\mathbf{v}\wedge\mathbf{n}) &= \tfrac{1}{2}\Delta v\mathbf{nw}.\mathbf{n}+\tfrac{1}{2}\Delta\mathbf{w}.\mathbf{nv}-\tfrac{1}{2}\Delta\mathbf{w}.\mathbf{vn}\\ &= \tfrac{1}{2}\Delta v\mathbf{nv}.\boldsymbol{\Phi}.\mathbf{n}-\tfrac{1}{2}\Delta v\mathbf{n}.\boldsymbol{\Phi}.\mathbf{vn}=0,\end{aligned} \tag{54}$$

since $\boldsymbol{\Phi}$ is self-conjugate, $\Delta\mathbf{w}=\Delta\mathbf{vn}.\boldsymbol{\Phi}$, and $\Delta\mathbf{w}.\mathbf{n}=0$. Thus

$$\int_S (\mathbf{vw}.\mathbf{n}-\tfrac{1}{2}\mathbf{v}.\mathbf{wn})\,dS=0 \tag{55}$$

even if S encloses characteristic surfaces on which $\mathbf{v}$ is discontinuous. This is the first integral identity.

The second integral identity is obtained from the first by putting $\mathbf{v}$ successively equal to $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_1-\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ both satisfy the linearized equations of flow, and subtracting the two identities. Since $\mathbf{v}_1.\mathbf{w}_2=\mathbf{v}_1.\boldsymbol{\Phi}.\mathbf{v}_2=\mathbf{v}_2.\mathbf{w}_1$ this gives

$$\int_S (\mathbf{v}_1\mathbf{w}_2.\mathbf{n}+\mathbf{v}_2\mathbf{w}_1.\mathbf{n}-\mathbf{v}_1.\mathbf{w}_2\mathbf{n})\,dS=0, \tag{56}$$

which is a relation between two different flows. In supersonic flows, $\mathbf{v}_1$ and $\mathbf{v}_2$ may be discontinuous on characteristic surfaces inside S without affecting the validity of the result.

6. Uniqueness

There is, at most, one solution of the linearized equations (16), (17), (18), *and* (19) *of section* 2 *which satisfies the boundary conditions* (i) *to* (ix) *of section* 3.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any two solutions, different if this be possible, and let $\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2$. Then $\mathbf{v}$ satisfies the linearized equations (16) to (19) and the boundary conditions $\mathbf{n}.\mathbf{v}=0$ on the mean body surface Σ, $\mathbf{U}.\mathbf{v}=0$ on mean free boundaries, and the conditions (iii) to (ix) of section 3.

Consider the identity obtained by multiplying (55) scalarly by $\mathbf{U}$, with $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$,

$$\int_S (\mathbf{U}.\mathbf{v}\mathbf{w}.\mathbf{n} - \tfrac{1}{2}\mathbf{v}.\mathbf{w}\mathbf{U}.\mathbf{n})\, dS = 0. \tag{57}$$

The proof is now in two sections, (i) for subsonic flow ($M < 1$), and (ii) for supersonic flow ($M > 1$).

(i) *Subsonic flow*

Take the closed surface S in equation (57) to consist of five parts:

Σ = both sides of the mean body surface plus small tubular surfaces surrounding its edges,

σ = the surface of the vortex sheet plus small cylindrical surfaces surrounding its edges,

S_1 = a cylindrical surface of large radius R_1 whose axis is parallel to $\mathbf{U}$ and passes near to the body,

S_2 = a plane $\mathbf{R}.\mathbf{U} = -R_1 U$ at a large distance R_1 upstream from the body,

S_3 = a plane $\mathbf{R}.\mathbf{U} = R_1 U$ at a large distance R_1 downstream from the body.

On Σ, $\mathbf{w}.\mathbf{n} = \mathbf{v}.\mathbf{n} = 0$, and $\mathbf{U}.\mathbf{n} = 0$ on the mean body surface, and the contribution from the tubular surfaces is non-zero only on the leading edges from boundary condition (iii). By using the result (45), the contribution from Σ is

$$-U \int \frac{\pi K^2 |\cos\psi\, ds|}{(1 - M^2 \sin^2\psi)^{\frac{1}{2}}}, \tag{58}$$

the integration being over the leading edge.

The integral over σ vanishes since the contributions from the opposite sides of the vortex sheet cancel, and the contribution to the momentum integral from the cylinder at the edges is perpendicular to $\mathbf{U}$.

The integrals over S_1 and S_2 are $O(1/R_1^2)$ as $R_1 \to \infty$ by the boundary condition (vii), and, from the condition (viii), the integral over S_3 is

$$-\frac{U}{2} \int_{S_3} (\phi_y^2 + \phi_z^2)\, dS + O(1/R_1^2) \tag{59}$$

as $R_1 \to \infty$. On proceeding to the limit $R_1 \to \infty$ the identity (57) gives

$$-U \int \frac{\pi K^2 |\cos\psi\, ds|}{(1 - M^2 \sin^2\psi)^{\frac{1}{2}}} - \tfrac{1}{2} U \int_{S_3} (\phi_y^2 + \phi_z^2)\, dS = 0. \tag{60}$$

Both the integrals in (60) converge by the boundary condition (ix), and since the integrands are essentially positive,

$$K = 0, \quad \text{and} \quad \phi_y = \phi_z = 0 \quad \text{on} \quad S_3. \tag{61}$$

Therefore $\mathbf{v}$ is continuous on S_3 and there is no vortex sheet in the difference flow, so that $\mathbf{v}$ is continuous and bounded everywhere except possibly at leading edges where $v \sim o(r^{-\frac{1}{2}})$. An application of the divergence theorem for $\phi\mathbf{w}$ with $\nabla.\mathbf{w} = 0$, namely,

$$\int \phi \mathbf{w}.\mathbf{n}\, dS = \int \mathbf{v}.\mathbf{w}\, d\tau, \tag{62}$$

to the space between the body and a sphere of very large radius now proves that $\mathbf{v} = 0$ everywhere in the same way as in the usual proof of the uniqueness theorem for incompressible flow (cf. **14**), and the uniqueness theorem is proved for the subsonic case.

The presence of free boundaries only modifies the proof very slightly, and it is necessary to take as part of the closed surface the mean surface of the free boundary. The contribution to the integral of equation (57) vanishes since $\mathbf{U}.\mathbf{v}$ and $\mathbf{U}.\mathbf{n}$ both vanish on this surface, and the proof then follows the same lines as before.

(ii) *Supersonic flow*

Take the closed surface to consist of Σ, σ, and S_3, as in the subsonic case, plus a closing characteristic surface S_4 which is the envelope of the downstream characteristic semi-cones with vertices on the leading supersonic edge of the mean body surface. The contributions from Σ and σ are the same as before, and the contribution from S_3 for finite R_1, not necessarily large here, is

$$-\tfrac{1}{2}U \int_{S_3} \{(M^2-1)\phi_x^2+\phi_y^2+\phi_z^2\}\, dS. \tag{63}$$

The contribution from S_4 vanishes, since $\mathbf{v} = 0$ just upstream from S_4 and the integrand in (57) has been proved to be continuous at characteristic surfaces in section 5. Thus the identity (57) gives

$$-U \int \frac{\pi K^2 |\cos\psi\, ds|}{(1-M^2\sin^2\psi)^{\frac{1}{2}}} - \tfrac{1}{2}U \int \{(M^2-1)\phi_x^2+\phi_y^2+\phi_z^2\}\, dS = 0, \tag{64}$$

the integration being over the leading subsonic edge in the first integral. Again both these integrals converge by the boundary condition (ix), and, since the integrands are essentially positive,

$$K = 0, \quad \text{and} \quad \phi_x = \phi_y = \phi_z = 0 \quad \text{on} \quad S_3. \tag{65}$$

The latter result is true for any S_3 which lies downstream from the body so that ϕ is constant over the whole of space downstream from any S_3, and all the derivatives of ϕ vanish in this space. The uniqueness theorem for a properly-set Cauchy initial value problem for the space upstream from S_3 (cf. **15**) now gives that $\mathbf{v} = 0$ upstream from S_3 and hence $\mathbf{v} = 0$ everywhere, which proves the uniqueness theorem for the supersonic case.

The modifications necessary when there is a free surface are the same as for the subsonic case.

7. A general reversed flow theorem

Let $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *be vector functions of position on the surfaces of two thin bodies having the same mean surface* Σ, *the two bodies being respectively in opposite flows for which the undisturbed stream velocities are* $\mathbf{U}$ *and* $-\mathbf{U}$. *Let* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *be perturbation velocities satisfying the linearized equations* (17), (18), *and* (19) *and satisfying the boundary conditions*

$$\mathbf{n}.\mathbf{v}_1 = -\mathbf{U}.\mathbf{n}_1, \qquad \mathbf{n}.\mathbf{v}_2 = \mathbf{U}.\mathbf{n}_2 \tag{66}$$

on Σ, *and otherwise satisfying the boundary conditions* (iii) *to* (ix) *of section* 3. *Let* p_1 *and* p_2 *be the corresponding linearized perturbation pressures,*

$$p_1 = -\rho_0 \mathbf{U}.\mathbf{v}_1, \qquad p_2 = \rho_0 \mathbf{U}.\mathbf{v}_2. \tag{67}$$

Then

$$\mathbf{U}.\int_\Sigma (p_1\mathbf{n}_2 + p_2\mathbf{n}_1)\, dS = 0. \tag{68}$$

The reversed flow theorem in this form is too general for its significance to be appreciated immediately, and some special cases of more immediate practical application are derived in the next section.

In order to prove this theorem, the identity (56), multiplied scalarly by $\mathbf{U}$, namely,

$$\int (\mathbf{U}.\mathbf{v}_1\,\mathbf{w}_2.\mathbf{n} + \mathbf{U}.\mathbf{v}_2\,\mathbf{w}_1.\mathbf{n} - \mathbf{v}_1.\mathbf{w}_2\,\mathbf{U}.\mathbf{n})\, dS = 0 \tag{69}$$

is applied to the closed surface which consists of the mean surface Σ, the surfaces of any vortex sheets, the surface at infinity, and tubular surfaces of very small radius surrounding the edges of the surface Σ and the edges of the vortex sheet where v_1 or v_2 may be infinite.

It is a simple consequence of the boundary conditions (iv) to (xiii) that the integrals over the surface of the vortex sheet and the surface at infinity vanish. The integrals over the small tubular surfaces at the edges of Σ also vanish since where v_1 is infinite, v_2 is bounded, by the boundary condition (iii), and vice versa. The integrals over the tubular surfaces surrounding the edges of the vortex sheet also vanish; for, if the edges in the two flows do not coincide over any part of their lengths, then where v_1 is infinite, v_2 is bounded and vice versa; if this be not the case, then any contribution from the integral in (56) is a vector normal to $\mathbf{U}$, and vanishes on being multiplied scalarly by $\mathbf{U}$.

Thus the only contribution to the integral in (69) is from Σ, and hence

$$\int_\Sigma (\mathbf{U}.\mathbf{v}_1\,\mathbf{w}_2.\mathbf{n} + \mathbf{U}.\mathbf{v}_2\,\mathbf{w}_1.\mathbf{n} - \mathbf{v}_1.\mathbf{w}_2\,\mathbf{U}.\mathbf{n})\, dS = 0. \tag{70}$$

On Σ, $\mathbf{U}.\mathbf{n} = 0$ by definition of Σ, and $\mathbf{w}_1.\mathbf{n} = \mathbf{v}_1.\mathbf{n}$ and $\mathbf{w}_2.\mathbf{n} = \mathbf{v}_2.\mathbf{n}$ so that, on multiplying (70) by ρ_0 and by using the relations (66) and (67), the relation (68) is proved.

It is to be noticed that p_1 and p_2 are the linearized perturbation pressures, so that any aerodynamic forces and moments on the respective bodies which are calculated on application of (68) do not include suction forces on sharp edges, as has been pointed out at the end of section 4. The theorem is still true if the flows include free boundaries on which $\mathbf{U}.\mathbf{v}_1$ and $\mathbf{U}.\mathbf{v}_2$ vanish.

8. Some special cases of the general reversed flow theorem

I. The simplest case occurs when the two bodies are the same and $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit normals to the body surface, so that $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}^*$. Then (68) becomes

$$\mathbf{U}.\int_{\Sigma} p_1 \mathbf{n}^* \, dS = -\mathbf{U}.\int_{\Sigma} p_2 \mathbf{n}^* \, dS, \tag{71}$$

and comparison with (51) shows that this result implies that the drag forces (neglecting leading edge suction) are of the same magnitude in the two flows. When Σ is plane this result was first given for supersonic flow in (**3**) and (**4**).

II. With the same bodies, as in **I**, let a portion Σ_1^* of the body surface† in the flow be given an extra small constant vector incidence $\boldsymbol{\alpha}_1$. Then if $\mathbf{n}^*$ was the normal to the original body surface, the new normal is approximately $\mathbf{n}^* + \boldsymbol{\alpha}_1 \wedge \mathbf{n}$, neglecting second-order quantities. Take now

$$\mathbf{n}_1 = \boldsymbol{\alpha}_1 \wedge \mathbf{n} \text{ on } \Sigma_1, \qquad \mathbf{n}_1 = 0 \text{ on the rest of } \Sigma, \qquad \mathbf{n}_2 = \mathbf{n}^* \text{ on } \Sigma; \tag{72}$$

then p_1 is the excess pressure consequent upon the change in the body surface, and (68) gives

$$\mathbf{U}.\int_{\Sigma} p_1 \mathbf{n}^* \, dS = -\mathbf{U}.\int_{\Sigma_1} p_2 \boldsymbol{\alpha}_1 \wedge \mathbf{n} \, dS,$$

or

$$\mathbf{U}.\int_{\Sigma} p_1 \mathbf{n}^* \, dS = -\mathbf{U} \wedge \boldsymbol{\alpha}_1 . \int_{\Sigma_1} p_2 \mathbf{n} \, dS. \tag{73}$$

On comparison with (49) and (51) it is seen that the left-hand side of (73) is proportional to the extra drag force (neglecting edge suction force) on the original body due to the extra incidence of Σ_1^*, and the right-hand side is proportional to the component perpendicular to $\mathbf{U}$ and in the plane of incidence of the original pressure in the reversed flow acting on Σ_1^*.

III. Again with the same bodies, let a portion Σ_1^* of the body surface be given a small angular velocity $\boldsymbol{\Omega}_1$ about a point whose position vector relative to some origin of vectors is $\mathbf{R}_1$. In the usual theory of such quasi-steady motions, it is assumed that the flow and forces are given correctly in linearized theory by warping the body suitably and considering the

† The reader is reminded that Σ^* denotes the actual body surface and Σ denotes the corresponding mean body surface on which the boundary conditions are applied (see section 4).

steady flow past the warped body. The boundary condition on the body surface at $\mathbf{R}$ is

$$\mathbf{n}^*.(\mathbf{U}+\mathbf{v}) = \mathbf{n}^*.\boldsymbol{\Omega}_1 \wedge (\mathbf{R}-\mathbf{R}_1), \tag{74}$$

or, on the mean body surface Σ, by neglecting second-order quantities,

$$\mathbf{n}.\mathbf{v} = -\mathbf{U}.\mathbf{n}^*+\mathbf{n}.\boldsymbol{\Omega}_1 \wedge (\mathbf{R}-\mathbf{R}_1). \tag{75}$$

In this case, take

$$\begin{aligned} \mathbf{n}.\mathbf{v}_1 = -\mathbf{U}.\mathbf{n}_1 = \mathbf{n}.\boldsymbol{\Omega}_1 \wedge (\mathbf{R}-\mathbf{R}_1) \quad &\text{on } \Sigma_1, \\ \mathbf{n}_1 = 0 \quad &\text{on the rest of } \Sigma, \\ \mathbf{n}_2 = \mathbf{n}^* \quad &\text{on } \Sigma; \end{aligned} \tag{76}$$

equation (68) then gives

$$\mathbf{U}.\int_{\Sigma} p_1 \mathbf{n}^* \, dS = \int_{\Sigma_1} p_2 \mathbf{n}.\boldsymbol{\Omega}_1 \wedge (\mathbf{R}-\mathbf{R}_1) \, dS,$$

or

$$\mathbf{U}.\int_{\Sigma} p_1 \mathbf{n}^* \, dS = \boldsymbol{\Omega}_1.\int_{\Sigma_1} p_2(\mathbf{R}-\mathbf{R}_1) \wedge \mathbf{n} \, dS. \tag{77}$$

On comparison with (50) and (51), it is seen that the left-hand side of (77) is proportional to the extra drag force on the original body caused by the angular motion of Σ_1^* and the right-hand side is proportional to the component in the direction of the axis of rotation of the moment of the original (reversed flow) pressures on Σ_1^* about the point $\mathbf{R}_1$.

IV. With any two bodies having the same mean surface Σ, let a part Σ_1^* of the body surface in the flow $\mathbf{U}$ be given a small extra vector incidence $\boldsymbol{\alpha}_1$ and a part Σ_2^* of the body surface in the flow $-\mathbf{U}$ be given a small vector incidence $\boldsymbol{\alpha}_2$, and take (cf. case **II**)

$$\begin{aligned} \mathbf{n}_1 = \boldsymbol{\alpha}_1 \wedge \mathbf{n} \quad &\text{on } \Sigma_1, \\ \mathbf{n}_1 = 0 \quad &\text{on the rest of } \Sigma, \\ \mathbf{n}_2 = \boldsymbol{\alpha}_2 \wedge \mathbf{n} \quad &\text{on } \Sigma_2, \\ \mathbf{n}_2 = 0 \quad &\text{on the rest of } \Sigma. \end{aligned} \tag{78}$$

Then p_1 and p_2 are the pressure changes consequent upon the surface changes, and (68) gives

$$\mathbf{U}.\int_{\Sigma_2} p_1 \boldsymbol{\alpha}_2 \wedge \mathbf{n} \, dS = -\mathbf{U}.\int_{\Sigma_1} p_2 \boldsymbol{\alpha}_1 \wedge \mathbf{n} \, dS,$$

or

$$\mathbf{U} \wedge \boldsymbol{\alpha}_2.\int_{\Sigma_2} p_1 \mathbf{n} \, dS = -\mathbf{U} \wedge \boldsymbol{\alpha}_1.\int_{\Sigma_1} p_2 \mathbf{n} \, dS. \tag{79}$$

On comparison with (49), the left-hand side of (79) is seen to be proportional to the component in the plane of the incidence $\boldsymbol{\alpha}_2$ and perpendicular to $\mathbf{U}$ of the force on Σ_2 due to the extra pressures caused by the extra incidence $\boldsymbol{\alpha}_1$ of Σ_1, and similarly for the right-hand side.

The special case of (79), when $\Sigma_1 = \Sigma_2 = \Sigma$ and Σ is a plane, and when

$\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are equal vectors perpendicular to $\mathbf{U}$ and in the plane of Σ, gives the lift slope reversal theorem of (**9**).

V. As in **IV**, let the area Σ_1^* in the flow $\mathbf{U}$ be given a small extra incidence $\boldsymbol{\alpha}_1$, let the area Σ_2^* in the flow $-\mathbf{U}$ be given a small angular velocity $\boldsymbol{\Omega}_2$ about the point $\mathbf{R}_2$ and take

$$\begin{aligned} \mathbf{n}_1 &= \boldsymbol{\alpha}_1 \wedge \mathbf{n} \quad \text{on } \Sigma_1, \\ \mathbf{n}_1 &= 0 \quad \text{on the rest of } \Sigma, \\ \mathbf{n}.\mathbf{v}_1 = \mathbf{U}.\mathbf{n}_2 &= \mathbf{n}.\boldsymbol{\Omega}_2 \wedge (\mathbf{R}-\mathbf{R}_2) \quad \text{on } \Sigma_2 \\ \mathbf{n}_2 &= 0 \quad \text{on the rest of } \Sigma. \end{aligned} \tag{80}$$

The p_1 and p_2 are the pressure changes consequent upon the surface changes, and (68) gives

$$\int_{\Sigma_2} p_1 \mathbf{n}.\boldsymbol{\Omega}_2 \wedge (\mathbf{R}-\mathbf{R}_2)\, dS = -\mathbf{U}.\int_{\Sigma_1} p_2 \boldsymbol{\alpha}_1 \wedge \mathbf{n}\, dS,$$

or

$$\boldsymbol{\Omega}_2.\int p_1(\mathbf{R}-\mathbf{R}_2)\wedge \mathbf{n}\, dS = -\mathbf{U}\wedge\boldsymbol{\alpha}_1.\int_{\Sigma_1} p_2 \mathbf{n}\, dS. \tag{81}$$

Comparison with (49) and (50) give an interpretation of (81) similar to those of the previous cases. A special case of this result, as in case **IV**, is given in (**9**).

VI. Let the area Σ_1^* in the flow $\mathbf{U}$ be given a small angular velocity $\boldsymbol{\Omega}_1$ about the point $\mathbf{R}_1$, let the area Σ_2^* in the flow $-\mathbf{U}$ be given a small angular velocity $\boldsymbol{\Omega}_2$ about the point $\mathbf{R}_2$, and take $\mathbf{n}_1$ as in (76) and $\mathbf{n}_2$ as in (80). Then p_1 and p_2 are the pressure changes consequent upon the surface changes, and (68) gives

$$\boldsymbol{\Omega}_2.\int_{\Sigma_2} p_1(\mathbf{R}-\mathbf{R}_2)\wedge\mathbf{n}\, dS = \boldsymbol{\Omega}_1.\int_{\Sigma_1} p_2(\mathbf{R}-\mathbf{R}_1)\wedge\mathbf{n}\, dS, \tag{82}$$

for which the interpretation is similar to those given above. Again special cases of this result are given in (**9**).

VII. The six special cases of the general equation (68) given above are intended to be illustrative and not exhaustive; for instance the surface Σ_1^* in the flow $\mathbf{U}$ has been taken to have constant vector incidence in cases **II**, **IV**, and **V**, whereas it might have been divided into two parts, say, with incidences in opposite directions, thus giving an aileron-like movement; many other such cases can easily be envisaged. The theorems are true for flow in wind tunnels with either closed or open working sections of sufficient length.

It is worth pointing out that not all the special cases **I**–**VI** are independent and some can be deduced from the others by superposition, since the pressures p_1 and p_2 are linear functions of the fluid velocity. The more general cases mentioned in the previous paragraph can also be obtained by superposition of a suitable choice of **I**–**VI**.

9. A special reversed flow theorem

There is another reversed flow theorem of which a special case has been given in (**3**), and incorrectly in (**7**), which is not deducible from the general theorem of section 7.

When there are no vortex sheets in the flow so that the line integral in (41) for the aerodynamic force vanishes, there is left the integral over S^* which depends quadratically on the perturbation velocities, so that the component of force perpendicular to **U** is of second order (the drag force is always of the second order, of course). If in addition there are no infinite velocities in the fluid, then the surface S^* may be taken to be the mean body surface Σ, so that

$$\mathbf{F} = \rho_0 \int_{\Sigma} (\mathbf{vw}.\mathbf{n} - \tfrac{1}{2}\mathbf{v}.\mathbf{wn})\, dS. \tag{83}$$

Now take the mean body surface Σ to consist of planes parallel to $z = 0$, say. The lateral component of force perpendicular to **U** and parallel to the mean surface Σ is

$$F_y = \rho_0 \int_{\Sigma} \phi_y \mathbf{w}.\mathbf{n}\, dS. \tag{84}$$

The y-component of the vector integral identity (56) taken over a closed surface which consists of Σ and the surface at infinity gives

$$\int_{\Sigma} (\phi_{1y} \mathbf{w}_2.\mathbf{n} + \phi_{2y} \mathbf{w}_1.\mathbf{n})\, dS = 0, \tag{85}$$

the integral over the surface at infinity vanishing by the boundary conditions (vii) to (ix). Since the boundary conditions on Σ for the flows **U** and $-\mathbf{U}$ give

$$\mathbf{n}.\mathbf{w}_1 = \mathbf{n}.\mathbf{v}_1 = -\mathbf{U}.\mathbf{n}^* = -\mathbf{n}.\mathbf{v}_2 = -\mathbf{n}.\mathbf{w}_2, \tag{86}$$

equation (85) becomes

$$\int_{\Sigma} (\phi_{1y} \mathbf{w}_1.\mathbf{n} + \phi_{2y} \mathbf{w}_2.\mathbf{n})\, dS = 0 \tag{87}$$

or, from (84),

$$F_{1y} = -F_{2y}. \tag{88}$$

Thus the y-component of force is reversed in direction and is unchanged in magnitude. This result is trivial for subsonic flow, since all the forces are zero.

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IMPROVEMENTS TO THE ACCURACY OF ARITHMETICAL SOLUTIONS TO CERTAIN TWO-DIMENSIONAL FIELD PROBLEMS

By L. C. WOODS
(*Engineering Department, The University, Oxford*)

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SUMMARY

An account is given of the methods available to improve the accuracy of arithmetical solutions on a square mesh for two-dimensional partial differential equations. 'Squaring' (**1**) and 'Relaxation' (**5**) depend for their accuracy on the fineness of the mesh on which the finite difference equations replace the differential equations. The same accuracy can be obtained on a coarse mesh, with less labour, by the use of more accurate difference formulae, rather than by refining the mesh. There are two ways of doing this:

(*a*) by using a higher order difference equation as a closer approximation to the differential equation, or

(*b*) by calculating a correction term from the higher differences to apply to the results obtained by using the first approximation.

A new method of calculating the correction term is developed.

1. Introduction

This paper describes two methods of improving the accuracy of arithmetical solutions obtained by 'Squaring' (**1, 2**) or 'Relaxation' (**5**), to elliptic partial differential equations on a square mesh. The types of equations considered in this paper are

$$(a)\ \nabla^2 f = \text{constant}, \qquad (b)\ \nabla^2 f = \phi(x,y), \qquad (c)\ \nabla^4 f = \phi(x,y)$$

$$(d)\ \nabla^2 f = \phi(f, f_x, f_y, x, y),$$

where suffixes denote derivatives. The arithmetical solutions are usually found by using the first finite difference approximations to the differential equations. The values of some wanted function at the centres of the squares of the mesh are expressed in terms of the four values at the corners of the squares. The accuracy of this first approximation can be improved by either using a higher order difference equation as a closer approximation to the differential equation (**4**), or calculating a correction term from the higher differences to apply to the results obtained using the first approximation (**3**).

The first method results in formulae expressing the value of the function at the centre of mesh points covering several squares. Bickley's term 'molecules' (**4**) will be used occasionally to describe the square or sets of

squares over which the finite difference equation spreads. This paper gives the large molecule formulae of Thom and Bickley, with some extensions.

In the second method two ways are available for calculating the error terms. For example, the largest error term, usually the only one of any importance can be obtained by

(*a*) calculating the expression $\Delta_0 = \frac{1}{12}(\delta^4_{ox}+\delta^4_{oy})$, where δ^4_{ox}, δ^4_{oy} are the fourth differences of the function f at the point O, in the OX and OY directions respectively; this is Fox's method.

(*b*) calculate instead $\Delta_0 \doteq \frac{a^4}{12}\left(\nabla^4 f - 2\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)$, and inserting the value of $\nabla^4 f$ from the field equation, e.g. for Laplace's $\nabla^4 f = 0$ and $\Delta_0 \doteq -\frac{a^4}{6}\frac{\partial^4 f}{\partial x^2 \partial y^2}$,

where a is the mesh size. The second of these two methods, which involves the use of the field equation in the calculation of the error term, requires a spread of only three grid points in each direction, as against five in Fox's method. The method, which is developed in section 4, proves easier to apply numerically, avoids extrapolation near the boundaries, an undesirable feature of Fox's method, and provides a criterion of suitable mesh size.

2. The finite difference formulae

Adopting and extending Bickley's notation (**4**) we write

$$S_1 = f_1+f_2+f_3+f_4 = \sum_1^4 f_i, \qquad S_2 = \sum_5^8 f_i, \qquad S_3 = \sum_9^{12} f_i,$$

$$S_4 = \sum_{13}^{20} f_i, \qquad S_5 = \sum_{21}^{24} f_i, \qquad \mathscr{D}^2 = \frac{\partial^2}{\partial x \partial y},$$

where f_i ($i = 1, 2, \ldots$) are the values of a wanted function at the lattice points shown in Fig. 1. Bickley then shows that

$$S_1 = \left\{4+a^2\nabla^2+\frac{2a^4}{4!}(\nabla^4-2\mathscr{D}^4)+\frac{2a^6}{6!}(\nabla^6-3\mathscr{D}^4\nabla^2)+ \right.$$
$$\left. +\frac{2a^8}{8!}(\nabla^8-4\mathscr{D}^4\nabla^4+2\mathscr{D}^8)+\ldots\right\} f_0, \quad (1)$$

$$S_2 = \left\{4+2a^2\nabla^2+\frac{4a^4}{4!}(\nabla^4+4\mathscr{D}^4)+\frac{4a^6}{6!}(\nabla^6+12\mathscr{D}^4\nabla^2)+ \right.$$
$$\left. +\frac{4a^8}{8!}(\nabla^8+24\mathscr{D}^4\nabla^4+16\mathscr{D}^8)+\ldots\right\} f_0, \quad (2)$$

$$S_3 = \left\{4+4a^2\nabla^2+\frac{32a^4}{4!}(\nabla^4-2\mathscr{D}^4)+\frac{128a^6}{6!}(\nabla^6-3\mathscr{D}^4\nabla^2)+ \right.$$
$$\left. +\frac{512a^8}{8!}(\nabla^8-4\mathscr{D}^4\nabla^4+2\mathscr{D}^8)+\ldots\right\} f_0; \quad (3)$$

and similarly it can be shown that

$$S_4 = \left\{8+10a^2\nabla^2+\frac{4a^4}{4!}(17\nabla^4+14\mathscr{D}^4)+\frac{4a^6}{6!}(65\nabla^6+105\mathscr{D}^6)+\right.$$
$$\left.+\frac{4a^8}{8!}(257\nabla^8+876\mathscr{D}^4\nabla^4-1054\mathscr{D}^8)+\ldots\right\}f_0, \quad (4)$$

$$S_5 = \left\{4+8a^2\nabla^2+\frac{64a^4}{4!}(\nabla^4+4\mathscr{D}^4)+\frac{256a^6}{6!}(\nabla^6+12\mathscr{D}^4\nabla^2)+\right.$$
$$\left.+\frac{1024a^8}{8!}(\nabla^8+24\mathscr{D}^4\nabla^4+16\mathscr{D}^8)+\ldots\right\}f_0. \quad (5)$$

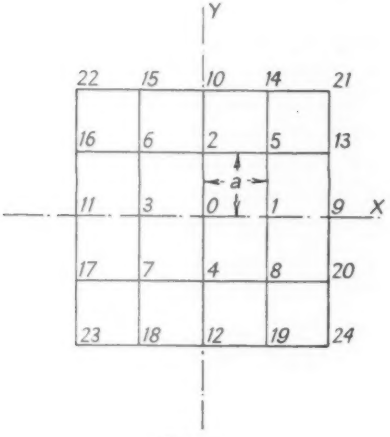

FIG. 1.

In the special case $\nabla^2 f = \text{constant} = k$ say, then $\nabla^4 f = \nabla^6 f = 0$, etc., and $\mathscr{D}^2\nabla^2 f = 0$, etc., and so (1) becomes

$$S_1 = 4f_0+a^2k-\frac{a^4}{6}\mathscr{D}^4 f_0+\frac{4a^8}{8!}\mathscr{D}^8 f_0+O(a^{12}), \quad (6)$$

from which follows the well-known first approximation

$$4f_0 \doteqdot -a^2k+\sum_1^4 f_i. \quad (7)$$

Equation (7) was employed by Thom (**1**) in 1928, in his method of 'squaring a field', and by Southwell (**5**) in 1938 in the related process termed 'relaxation'.

The accuracy obtained by employing (7) on a given mesh can be improved by four distinct methods:—

(*a*) Reduce the mesh size a, which, from (6), obviously reduces the size of the error term. This, which is the normal procedure in relaxing or

squaring, is repeated until a subdivision does not alter the values of f found at the lattice points of the previous mesh.

(*b*) Use a larger molecule obtained by use of (1) and (2) to eliminate the $a^4\mathscr{D}^4$ error term (four squares); or use of (3), (4), and (5) to eliminate both the $a^4\mathscr{D}^4$ and $a^8\mathscr{D}^8$ terms (sixteen squares). All the ∇^2 error terms disappear, of course, as in (6).

(*c*) Calculate the largest error term using Fox's difference method, and correct the results for this error. From the general equation (1), the largest error term is

$$\frac{a^4}{12}(\nabla^4-2\mathscr{D}^4)f_0 = \frac{a^4}{12}\left(\frac{\partial^4 f_0}{\partial x^4}+\frac{\partial^4 f_0}{\partial y^4}\right) = \frac{1}{12}(\delta_{ox}^4+\delta_{oy}^4)+O(a^6),$$

but, as already mentioned, the calculation of the term in this form does not involve the field equation.

(*d*) If the field equation is used, in this case $\nabla^2 f = k$, then $\nabla^4 f = 0$, and the error term reduces to $-(a^4/6)\mathscr{D}^4 f$, as in (6).

The larger molecules used by Thom amount to a repeated application of (7) over a mesh of four or sixteen squares, carried out in one operation. The a^4 error terms are not eliminated, and thus it is essentially a 'speeding-up' process. However, accuracy is in fact improved, because the possible cumulative errors, arising in the application of (7) square by square to the mesh do not occur.

3. Large molecule formulae

(*a*) $\nabla^2 f = \text{constant} = k$.

Thom's formulae (**2**), with the addition of the largest error terms, are:

4 squares: $$12f_0 = 2S_1+S_2-4a^2k-(a^4/3)\mathscr{D}^4 f_0+O(a^8); \tag{8}$$

16 squares: $$476f_0 = 46S_3+32S_4+9S_5-576a^2k-48a^4\mathscr{D}^4 f_0+O(a^8). \tag{9}$$

Alternatively:

$$100f_0 = 10S_3+7S_4+S_5+118a^2k-\frac{22}{3}a^4\mathscr{D}^4 f_0+O(a^8), \tag{10}$$

an equation with convenient coefficients, and incidentally a lower error term than (9) which is based on (7). All these may be verified by adding (1) to (5), putting $\nabla^2 f = k$, in the correct proportion. They are particularly useful for rectangular fields with given boundary values.

Eliminating $a^4\mathscr{D}^4 f_0$ from (1) and (2) yields Bickley's 'four-square' formula

$$20f_0 = 4S_1+S_2-6a^2k-\frac{80}{8!}a^8\mathscr{D}^8 f_0+O(a^{12}). \tag{11}$$

In a similar way $a^4\mathscr{D}^4 f_0$ and $a^8\mathscr{D}^8 f_0$ can be eliminated from (3), (4), and (5) to give an expression with an error term $O(a^{12})$, but with very large and

impractical coefficients. The high accuracy of such an expression would be generally unwarranted.

(*b*) $\nabla^2 f = \phi(x, y)$.

In this case (1) becomes:

$$S_1 = 4f_0 + a^2\phi_0 + \frac{a^4}{12}(\nabla^2\phi_0 - 2\mathscr{D}^4 f_0) + \frac{2a^6}{6!}(\nabla^4\phi_0 - 3\mathscr{D}^4\phi_0) + O(a^8). \quad (12)$$

From the corresponding equations for S_3, S_4, and S_5 the a^4 term can be eliminated to give

$$12f_0 = 8S_4 - 9S_3 - 4S_5 - 12a^2\phi_0 + O(a^6). \quad (13)$$

If in addition $\nabla^4 f =$ constant, i.e. $\nabla^2\phi =$ constant $= B$, say, the a^4 and a^6 terms can be eliminated to give

$$1332f_0 = 84S_3 + 128S_4 - 7S_5 - 1560a^2\phi_0 - 456B + O(a^8), \quad (14)$$

a result of little practical value. Bickley's procedure implies that we should use the form corresponding to (11), i.e.

$$20f_0 = 4S_1 + S_2 - 6a^2\phi_0 - \tfrac{1}{2}a^4\nabla^4 f_0 + O(a^6), \quad (15)$$

which eliminates the $a^4\mathscr{D}^4$ term, to calculate an approximate f_0, by ignoring the $a^4\nabla^4$ term. This term is then found approximately from

$$a^4\nabla^4 f_0 = a^2(a^2\nabla^2\phi_0) \doteqdot a^2\Big(\sum_1^4 \phi_i - 4\phi_0\Big),$$

and applied as a correction term. Bickley does not state this directly, his work being expressed as if the computor desired to calculate the values of $\nabla^2 f$ from a given f distribution, rather than seeking an f distribution to satisfy a relationship involving $\nabla^2 f$. He also incorrectly states that the above procedure is superior to Fox's difference method, since 'it is two-dimensional in character'. Bickley has used a larger molecule to eliminate the $a^4\mathscr{D}^4$ term, and a correction term to allow for the $a^4\nabla^4$ term. This is equivalent to (13) which eliminates both terms. The difference method is exactly equivalent in its elimination of the largest error term, treating both the parts $a^4\mathscr{D}^4$ and $a^4\nabla^4$ as correction terms, since

$$\frac{1}{12}(\delta_{ox}^4 + \delta_{oy}^4) \doteqdot \frac{a^4}{12}\left(\frac{\partial^4 f_0}{\partial x^4} + \frac{\partial^4 f_0}{\partial y^4}\right) = \frac{a^4}{12}\nabla^4 f_0 - \frac{a^4}{6}\mathscr{D}^4 f_0.$$

(*c*) $\nabla^4 f = \phi(x, y)$.

The first approximation can be obtained by two methods. The first was introduced by Thom (**2**) in connexion with viscous flow. This consists in using an intermediate function $\zeta(x, y)$, so that $\nabla^2 f = \zeta$, and $\nabla^2\zeta = \phi(x, y)$. When the ζ field is settled (or relaxed) with approximate boundary conditions, the f field is settled from $\nabla^2 f = \zeta$; this alters the boundary conditions for the ζ field, which then needs to be recalculated. The fields

are settled alternately in this way until the change from one round to the next is negligible. The final accuracy can be improved by any of the methods tabulated in section 2.

Thom also established the fundamental equation of the second method, which can be obtained by using (1), (2), and (3) to eliminate $\nabla^2 f_0$ and $\mathscr{D}^4 f_0$:

$$20f_0 = 8S_1 - 2S_2 - S^3 + a^4\nabla^4 f_0 + \tfrac{1}{6}a^6\nabla^6 f_0 - \tfrac{1}{3}a^6\mathscr{D}^4\nabla^2 f_0 + O(a^8).$$

The error term, $\frac{1}{6}a^6(\nabla^6 f_0 - 2\mathscr{D}^4\nabla^2 f_0)$, could be computed by methods similar to those in section 4 below. Alternatively (4) could be used to eliminate this term, but the resulting molecule would be cumbersome.

(*d*) $\nabla^2 f = \phi(f, f_x, f_y, x, y)$.

For special simple cases of this general equation, larger molecules could be derived, but as the elementary molecules themselves are often complex enough, the practical procedure is to apply the correction term as described below.

4. The correction term method

When $\nabla^2 f = \phi(x, y)$ as in (12), then

$$4f_0 = \sum_1^4 f_i - a^2\phi_0 - \frac{a^4}{12}(\nabla^2\phi_0 - 2\mathscr{D}^4 f_0) - \frac{2a^6}{6!}(\nabla^4\phi_0 - 3\mathscr{D}^4\phi_0) + O(a^8), \tag{16}$$

which covers the types of equations (*a*) and (*b*) of section 1. From (1) and (2),

$$S_2 - 2S_1 + 4f_0 = a^4\mathscr{D}^4 f_0 + \frac{a^6}{12}\mathscr{D}^4\nabla^2 f_0 + \frac{a^8}{360}\mathscr{D}^4\nabla^4 f_0 + \frac{a^8}{720}\mathscr{D}^8 f_0 + O(a^{10}), \tag{17}$$

a result which holds irrespective of the value of $\nabla^2 f$. Hence

$$a^4\mathscr{D}^4 f_0 = S_2 - 2S_1 + 4f_0 - \frac{a^6}{12}\mathscr{D}^4\phi_0 - \frac{a^8}{360}\mathscr{D}^4\nabla^2\phi_0 - \frac{a^8}{720}\mathscr{D}^8 f_0 + O(a^{10}).$$

Also from (1) applied to ϕ,

$$a^2\nabla^2\phi_0 = \sum_1^4 \phi_i - 4\phi_0 - \frac{a^4}{12}(\nabla^4\phi_0 - 2\mathscr{D}^4\phi_0) + O(a^{10}).$$

Using these in (16) gives

$$4f_0 = \sum_1^4 f_i - a^2\phi_0 + \tfrac{1}{6}(S_2 - 2S_1 + 4f_0) - \frac{a^2}{12}\Big(\sum_1^4 \phi_i - 4\phi_0\Big) + \\ + \frac{a^6}{240}\nabla^4\phi_0 - \frac{7}{360}a^6\mathscr{D}^4\phi_0 + O(a^8) \tag{18}$$

or in the special case $\phi = 0$, or a constant

$$4f_0 = \sum_1^4 f_i - a^2\phi_0 + \tfrac{1}{6}(S_2 - 2S_1 + 4f_0) - \frac{a^8}{3024}\mathscr{D}^8 f_0 + O(a^{12}). \tag{19}$$

(18) and (19) will be written in the form

$$4f_0 = \sum_1^4 f_i - a^2\phi_0 + \Delta_0 + \delta_0 + O(a^8), \tag{20}$$

where

$$\Delta_0 = \tfrac{1}{6}(S_2 - 2S_1 + 4f_0) - \frac{a^2}{12}\Big(\sum_1^4 \phi_i - 4\phi_0\Big), \tag{21}$$

and

$$\delta_0 = \frac{a^6}{240}\nabla^4\phi_0 - \frac{7}{360}a^6\mathscr{D}^4\phi_0, \tag{22}$$

or if $\phi = 0$, or a constant,

$$\Delta_0 = \tfrac{1}{6}(S_2 - 2S_1 + 4f_0), \tag{23}$$

$$\delta_0 = -\frac{a^8}{3024}\mathscr{D}^8 f_0. \tag{24}$$

Ignoring the secondary error term δ_0, it is required to solve

$$4f_0 = \sum_1^4 f_i - a^2\phi_0 + \Delta_0.$$

If f_0' is a first approximation satisfying $4f_0' = \sum_1^4 f_i' - a^2\phi_0$, Δ_0' is calculated from (21), i.e.

$$\Delta_0' = \tfrac{1}{6}\Big(\sum_5^8 f_i' - 2\sum_1^4 f_i' + 4f_0'\Big) - \frac{a^2}{12}\Big(\sum_1^4 \phi_i - 4\phi_0\Big).$$

The second approximation is found from $4f_0'' = \sum_1^4 f_i'' - a^2\phi_0 + \Delta_0'$, or putting $f_i'' = f_i' + g_i'$, the equation $4g_0' = \sum_1^4 g_i' + \Delta_0'$ is solved in the usual manner. We now put $\Delta_0'' = \tfrac{1}{6}\Big(\sum_5^8 g_i' - 2\sum_1^4 g_i' + 4g_0'\Big)$, and find $4g_0'' = \sum_1^4 g_i'' + \Delta_0''$, etc., so that finally, $f_i = f_i' + g_i' + g_i'' + \ldots$, and $\Delta_i = \Delta_i' + \Delta_i'' + \ldots$. The boundary values for f_i' are the same as those for f_i, while $g_i' = g_i'' = \ldots = 0$ on these boundaries. The method converges rapidly and often it is unnecessary to proceed beyond g_i'. The method differs from that of Fox by the way in which Δ_i is obtained.

Equation (23) can be written

$$6\Delta_0 = \Big(\sum_5^8 f_i - \sum_1^4 f_i\Big) - \Big(\sum_1^4 f_i - 4f_0\Big) \tag{25}$$

$$= \Big(\sum_5^8 f_i - \sum_1^4 f_i\Big) - R_0,$$

where 'R_0' is the 'residual' of the relaxation process. If residuals are known then the calculation of Δ_0 is simplified. The author's method is to

check the residuals at the end of the usual relaxation process, and to compute $6\Delta_0$ at the same time by setting up the difference table:

$$
\begin{array}{lll}
4f_0 & & \\
 & \left(\sum_1^4 f_i - 4f_0\right) & \\
\sum_1^4 f_i & & \left(\sum_5^8 f_i - \sum_1^4 f_i\right) - \left(\sum_1^4 f_i - 4f_0\right) \quad (= 6\Delta_0) \\
 & \left(\sum_5^8 f_i - \sum_1^4 f_i\right) & \\
\sum_5^8 f_i. & &
\end{array}
$$

Assuming that the residuals need checking in any case, this method of calculating Δ_0 is quicker than Fox's method of working out the fourth differences, particularly when ϕ is zero or a constant. If δ_0 is required, it can easily be obtained by noting that $a^6\nabla^4\phi_0 \doteqdot a^2\left(\sum_1^4 a^2\nabla^2\phi_i - 4a^2\nabla^2\phi_0\right)$, and $a^6\mathscr{D}^4\phi_0 \doteqdot a^2\left(\sum_5^8 \phi_i - 2\sum_1^4 \phi_i + 4\phi_0\right)$. $a^2\nabla^2\phi_i$ is computed when finding Δ_0. If ϕ = constant,

$$\delta_0 = -\frac{a^8}{3024}\mathscr{D}^8 f_0, \text{ and } a^8\mathscr{D}^8 f_0 = a^4\mathscr{D}^4(a^4\mathscr{D}^4)f_0 \doteqdot 6\left(\sum_5^8 \Delta_i - 2\sum_1^4 \Delta_i + 4\Delta_0\right). \tag{26}$$

5. Boundary conditions

In this section methods are given for calculating Δ_0 on the boundaries on which are specified (*a*) boundary values, (*b*) normal gradients.

(*a*) *Values specified*

Since
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \phi(x, y),$$

then
$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial x^2}\left(\phi - \frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^4 f}{\partial x^4}. \tag{27}$$

Similarly
$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^4 f}{\partial y^4}. \tag{28}$$

These equations give the values of $\mathscr{D}^4 f$ along boundaries x = constant and y = constant. For irregular boundaries (5), values at 'fictitious nodes' can be obtained by extrapolation in the usual way, although it is often possible to avoid 'irregular stars' by a conformal transformation (**2**). This is, of course, not an algebraic transformation applicable only to special cases, but an arithmetical transformation applicable (at least in principle) to all shapes of regions. An example very suitable for this type of approach is the incompressible flow ($\nabla^2\eta = 0$) about an aerofoil. The calculations are carried out in the square mesh formed by the velocity equipotentials

(η = constant) and the streamlines (ζ = constant). The aerofoil boundary is a slit along $\eta = 0$. The boundary conditions, e say, are specified as functions of position s ($e = \epsilon(s)$) on the aerofoil, but from an assumed relation $s = s(\eta)$, we can find approximate boundary conditions for the (η, ζ) plane, $e = e(\eta)$, and hence a relaxation solution can be found in this plane. By an integration along the boundary this solution yields a new relation $s_1 = s_1(\eta)$, and hence a new relation $e_1 = e_1(\eta)$. Relaxation yields a new solution in the (η, ζ) plane, and hence a new relation $s_2 = s_2(\eta)$, and so on. This iterative process converges rapidly.

If ϕ = constant, or varies slowly,

$$\Delta_0 \doteq \frac{a^4}{6}\mathscr{D}^4 f_0 = -\frac{a^4}{6}\frac{\partial^4 f_0}{\partial x^4}.$$

i.e. $\Delta_0 \propto \delta^4_{ox}$ where δ^4_{ox} are the fourth differences along the boundary. Thus for constant accuracy, the mesh length a must be adjusted so that the fourth differences remain approximately constant along the boundary. This proves a useful guide for mesh size variation near the nose of an aerofoil, sharp corners, etc.

(*b*) *Normal gradients specified*

From Fig. 1

$$f_5 = f_0 + a\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)f_0 + \frac{a^2}{2!}\left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right)f_0 + \ldots, \text{ etc.},$$

and thus

$$f_5 + f_6 = 2f_0 + 2a\frac{\partial f_0}{\partial y} + a^2\left(\frac{\partial^2 f_0}{\partial x^2} + \frac{\partial^2 f_0}{\partial y^2}\right) + \frac{a^3}{3!}\left(6\frac{\partial^3 f_0}{\partial x^2 \partial y} + 2\frac{\partial^3 f_0}{\partial y^3}\right) + \ldots$$

$$= 2f_0 + 2a\frac{\partial f_0}{\partial y} + a^2\nabla^2 f_0 + a^3\frac{\partial^3 f_0}{\partial x^2 \partial y} + \frac{a^3}{3}\frac{\partial^3 f_0}{\partial y^3} + \frac{a^4}{12}(\nabla^4 f_0 + 4\mathscr{D}^4 f_0) + O(a^6). \tag{29}$$

Also $$f_1 + f_3 = 2f_0 + \frac{2}{2!}a^2\frac{\partial^2 f_0}{\partial x^2} + \frac{2a^4}{4!}\frac{\partial^4 f_0}{\partial x^4} + \frac{2a^6}{6!}\frac{\partial^6 f_0}{\partial x^6} + \ldots$$

and $$f_2 = f_0 + a\frac{\partial f_0}{\partial y} + \frac{a^2}{2!}\frac{\partial^2 f_0}{\partial y^2} + \frac{a^3}{3!}\frac{\partial^3 f_0}{\partial y^3} + \ldots.$$

Hence

$$f_1 + f_3 + 2f_2 - 4f_0 = 2a\frac{\partial f_0}{\partial y} + a^2\nabla^2 f_0 + \frac{a^3}{3}\frac{\partial^3 f_0}{\partial y^3} + \frac{a^4}{12}(\nabla^4 f_0 - 2\mathscr{D}^4 f_0) + O(a^6). \tag{30}$$

From (29) and (30), $f_1 + f_3 + 2f_2 - (f_5 + f_6) - 2f_0 \doteq -a^3\frac{\partial^3 f_0}{\partial x^2 \partial y} - \frac{a^4}{2}\mathscr{D}^4 f_0$, i.e. if $\frac{\partial^2 f_0}{\partial x^2} + \frac{\partial^2 f_0}{\partial y^2} = \phi_0$, equation (30) can be written

$$4f_0 \doteq f_1 + f_3 + 2f_2 - 2a\frac{\partial f_0}{\partial y} - a^2\phi_0 + \\ + \tfrac{1}{3}\{f_5 + f_6 - (f_1 + f_3 + 2f_2) + 2f_0\} - \frac{a^3}{3}\frac{\partial \phi_0}{\partial y} - \frac{a^4}{12}\nabla^2\phi_0, \tag{31}$$

which corresponds to (18). Thus in this case

$$\Delta_0 = \tfrac{1}{3}\{f_5+f_6-(f_1+f_3+2f_2)+2f_0\}-\frac{a^3}{3}\frac{\partial\phi_0}{\partial y}-\frac{a^4}{12}\nabla^2\phi_0,$$

the last two terms of which can be calculated when $\phi = \phi(x, y)$ is specified. If $\nabla^2 f = 0$, a large molecule formula corresponding to (11) can be found by substituting $\nabla^2 f = 0$ in (29) and (30). This leads to

$$f_5+f_6+2(f_1+f_3+2f_2)-10f_0-6a\frac{\partial f_0}{\partial y}+O(a^6). \tag{32}$$

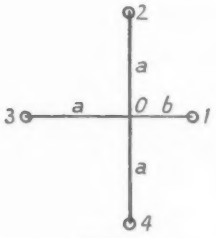

FIG. 2.

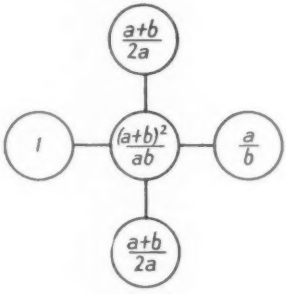

FIG. 3.

(c) *Irregular stars*

If irregular stars are encountered they can be handled by a method described by Southwell (5). The results, however, have an error term of $O(a^2)$, whereas the error term away from the boundaries is $O(a^4)$. Extrapolation formulae giving greater accuracy near the boundaries are unnecessarily complicated (3). The following method is relatively simple and reduces the error term on the boundary to $O(a^3)$.

From Fig. 2 we have

$$f_2+f_4 = 2f_0+a^2\frac{\partial^2 f_0}{\partial y^2}+O(a^4),$$

$$f_1 = f_0+b\frac{\partial f_0}{\partial x}+\frac{b^2}{2}\frac{\partial^2 f_0}{\partial x^2}+O(a^3),$$

$$f_3 = f_0-a\frac{\partial f_0}{\partial x}+\frac{a^2}{2}\frac{\partial^2 f_0}{\partial x^2}+O(a^3).$$

Hence $$f_2+f_4+\frac{a}{b}f_1+f_3 = \left(3+\frac{a}{b}\right)f_0+a^2\nabla^2 f_0+\frac{a}{2}(b-a)\phi_0+O(a^3),$$

from which follows

$$\frac{a+b}{2a}(f_2+f_4)+\frac{a}{b}f_1+f_3 = \frac{(a+b)^2}{ab}f_0+\frac{a}{2}(b-a)\phi_0+O(a^3). \tag{33}$$

Fig. 3 shows the resulting relaxation pattern for $\phi = 0$.

6. Advantages of the method

1. The calculation of fourth differences requires a spread of five lattice points in each direction to obtain one central value of the primary error term, which forces the computor to extrapolate near the boundaries. Equation (21) shows that the present method requires a spread of only three lattice points in each direction.

2. The large molecule formulae, e.g. (11), are very useful under certain conditions, but it is usually quicker to keep to the simplest relaxation pattern and use the correction term method, particularly when working through graded meshes.

3. It would be possible to calculate and include the second error term, δ_0, by the method described in section 4. The calculation of δ_0 by the use of sixth differences would be possible, but clumsy. However, near the boundaries the excessive extrapolation required would give doubtful results, a disadvantage not possessed by the author's method.

4. For a given number of significant figures a solution of (7) can be improved by either

(*a*) advancing to the next grade of mesh of side $a/2$ in the normal manner, or

(*b*) applying the first correction term as described in section 4, to the coarse mesh.

The question arises which of these results will have the greatest accuracy. The methods of Fox and Bickley give no direct answer to this question, which is certainly important if it is decided for some reason or other that results *are* required on a finer mesh. If, for instance, it can be determined that method (*b*) gives the more accurate results, then *during the advance* to the finer mesh these accurate results could be maintained constant, just as if they were fixed boundary values. This would obviously considerably reduce the work involved in the advance to the finer mesh. Of course, if the results on the finer mesh are *then* improved by use of the correction term, the values maintained constant during the advance must be permitted to change in the usual way.

If in (16) a is replaced by a/n, each mesh side is subdivided into n parts, and on the resulting finer mesh the first error term is Δ_0/n^4. If, on the other hand, Δ_0 is eliminated on the coarse mesh, the magnitude of the remaining error term is δ_0. When

$$\Delta_0/n^4 = \delta_0, \quad \text{i.e.} \quad n = (\Delta_0/\delta_0)^{\frac{1}{4}},$$

then n can be termed the 'equivalent mesh size reduction'. Such a mesh size reduction, in conjunction with the ordinary methods, is equivalent in accuracy to use of the correction term on the coarse mesh; δ_0 can be

calculated at a few points from equation (26). The values of n at these points will give an indication of the comparative accuracy of the correction term method. For instance, if $n = 2$, the values found using the correction term could be maintained constant during a reduction of the mesh length to $a/2$. This is illustrated in example (i) below.

Equations of the form $\nabla^4 f =$ constant are amenable to a similar treatment as can be seen from (16); but equations of the type $\nabla^2 f = \phi(f, f_x, f_y, x, y)$ are too complex to receive a general treatment. Simple cases, such as $\nabla^2 f + \lambda f = 0$, can be treated by the author's method as illustrated by example (ii) below.

7. Examples

The following two examples were treated by Fox and form a basis for comparison. In this section, as in the previous sections, subscript '0' will be used to denote a central value, subscript i ($i = 1, 2, \ldots, 4$) will indicate

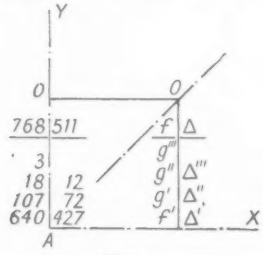

FIG. 4.

values on the four nearest mesh points, and i ($i = 5, \ldots, 8$) will indicate values at the next four mesh points.

(i) *Solution of* $\nabla^2 f = -2560$, *with* $f = 0$ *on* $x = \pm 1$, $y = \pm 1$.

By symmetry only one octant need be considered, although the figures are drawn for one quadrant for convenience. The first approximation is

$$4f_0' = \sum_1^4 f_i' + 2560a^2. \tag{34}$$

(*a*) $a = 1$. Equation (34) gives $f_0' = 640$ (see Fig. 4). Fox was unable to improve this value. From (23)

$$\Delta_0' = \frac{4 \times 640}{6} = 427, \quad \text{and thus} \quad 4g_0' = \sum_1^4 g_i' + \Delta_0' = 427, \quad \text{as} \quad g_i = 0.$$

Hence $g_0' = \dfrac{427}{4} = 107$. Again $\Delta_0'' = \frac{1}{6}\left(\sum_5^8 g_i' - 2\sum_1^4 g_i' + 4g_0'\right) = 72$.

Hence $g_0'' = \dfrac{72}{4} = 18$. $\qquad \Delta_0''' = \dfrac{4 \times 18}{6} = 12$.

Thus $\qquad g_0''' = 3$.

Thus $f_0 = f_0' + g_0' + g_0'' + g_0''' = 768$, which is 1·8 per cent. in error from the correct analytical result. A value of n can be calculated, for using (27) and (28), $\Delta_i = 0$ $(i = 1,\ldots, 8)$ on the boundary, and hence, from (26),

$$\delta_0 = -\frac{6}{3024}\left(\sum_5^8 \Delta_i - 2\sum_1^4 \Delta_i + 4\Delta_0\right) = -\frac{6}{3024}(4\times 511).$$

Hence $n = \left(\dfrac{\Delta_0}{\delta_0}\right)^{\frac{1}{4}} \doteqdot 3$ (taking the lower nearest integer).

Thus the value 768 is more accurate than that obtained by dividing the mesh length by 2, and employing (7). In fact, by using ordinary relaxation on a mesh $a = \frac{1}{2}$, the error is 4·7 per cent., and with $a = \frac{1}{4}$ the error is 1·2 per cent., which confirms the intermediate value of $n = 3$ corresponding to 1·8 per cent. error. Thus we are justified in retaining 768 as a fixed value during the advance to $a = \frac{1}{2}$.

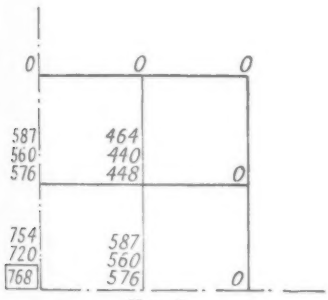

FIG. 5.

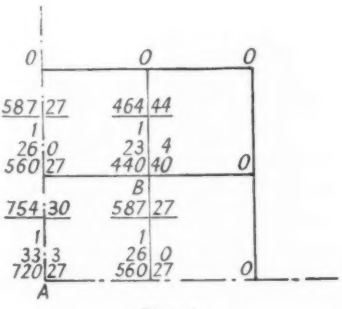

FIG. 6.

(*b*) $a = \frac{1}{2}$. Fig. 5 gives the results obtained using (7) on this mesh, with the central value fixed. The middle entries at each point are the results obtained by using ordinary methods (equation (7), without the central value fixed), while the top figures are the correct values. In Fig. 6 we have started with the results found by ordinary methods so that the values can be compared with those found by Fox.

Dealing with point B as an example:

$$\Delta_B' = \tfrac{1}{6}\{4\times 440 - 2(560+560+0+0) + (720+0+0+0)\} \doteqdot 40.$$

This calculation is repeated at each lattice point, the values so found being entered to the right of the f values. The value g_B' is found by the usual relaxation process, giving $g_B' = 23$, shown entered above 440. Again

$$\Delta_B'' = \tfrac{1}{6}\{4\times 23 - 2(0+26+26+0) + (33+0+0+0)\} \doteqdot 4,$$

which, together with the other Δ_i'' gives rise to $g_B'' = 1$. The final value of $f_i = f_i' + g_i' + \ldots$, $\Delta_i = \Delta_i' + \Delta_i'' + \ldots$, are shown above the line at each

point in Fig. 6. As before, values of n can be calculated and they are found to be $n_A \doteqdot 3{\cdot}7$, $n_B \doteqdot 3{\cdot}8$. Thus these results would be quite accurate up to $a \doteqdot 1/(2\times 3{\cdot}5) = \frac{1}{7}$.

(c) $a = \frac{1}{4}$. The results given in Fig. 6 are now entered on Fig. 7 (shown in squares) and maintained constant. Intermediate values are entered using (34), and the only large residues arising are shown in circles. Because

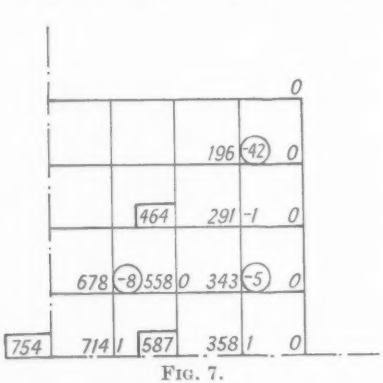

FIG. 7.

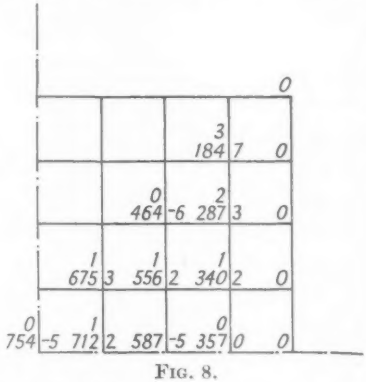

FIG. 8.

of these fixed values this field is easily relaxed and the results are given in Fig. 8. The values of Δ_i, plus small residuals remaining from the relaxation obtained from this field, are shown to the right of the lattice points in Fig. 8, and the corresponding values of g_i' are entered above the f_i'. The solution $f_i = f_i' + g_i'$ differs from the analytical solution quoted by Fox at the most by unity in each entry. At the centre n is now greater than 3, an indication that the accuracy on this mesh is at least as good as that obtained by ordinary methods on a mesh size $a = 1/12$.

(ii) *An Eigenvalue problem* $\quad \nabla^2 w + \lambda w = 0.$ (35)

If w is an approximate solution of (35), Rayleigh's Principle gives

$$\lambda \doteqdot -\frac{\iint w\nabla^2 w\, dxdy}{\iint w^2\, dxdy}. \tag{36}$$

From (16) $\quad a^2\nabla^2 w_0 = \sum_1^4 w_i - 4w_0 - \frac{a^4}{12}(\nabla^4 - 2\mathscr{D}^4)w_0 + O(a^6),$

but $\nabla^4 w = -\lambda\nabla^2 w = \lambda^2 w$, and thus

$$a^2\nabla^2 w_0 = \sum_1^4 w_i - 4w_0 - \frac{a^4}{12}(\lambda^2 w_0 - 2\mathscr{D}^4 w_0), \tag{37}$$

or, less accurately, $\quad a^2\nabla^2 w_0 = \sum_1^4 w_i - 4w_0.$ (38)

Starting with an approximate solution to the mode w, $\nabla^2 w$ is calculated from (38) and λ found from (36). This value of λ is used in (35) and from $a^2\lambda w_0 + \sum_1^4 w_i - 4w_0 = 0$, a new w field is calculated. From this a new λ is calculated, and so on, the process being repeated until no further improvement is possible.

In the example given by Fox the problem is to determine the fundamental vibration of a square membrane of unit side. The analytical solution is

$$w = \cos \pi x \cos \pi y, \qquad \lambda = 2\pi^2 \doteq 19{\cdot}74.$$

Fig. 9 shows one quadrant, with $a = \frac{1}{4}$, giving the first approximation to w, obtained by the process mentioned above. The values of $a^2\nabla^2 w$ found from (38) are shown to the right of the lattice points. The corresponding value of $a^2\lambda$ calculated from (36) is 1·172, i.e. $\lambda = 18{\cdot}75$. Then from (37)

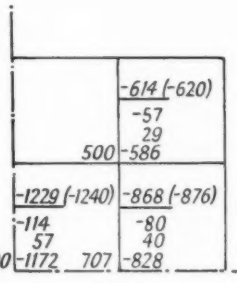

FIG. 9.

$$a^2\nabla^2 w = \sum_1^4 w_i - 4w_0 - 0{\cdot}115w_0 + \tfrac{1}{6}a^4\mathscr{D}^4 w_0.$$

The two correction terms, '$-0{\cdot}115w_0$' and '$\frac{1}{6}a^4\mathscr{D}^4 w_0$', are easily found and are shown in Fig. 9 above the approximate values of $a^2\nabla^2 w$. Using these new values of $a^2\nabla^2 w_i$ in (36) we find $a^2\lambda = 1{\cdot}226$, therefore $\lambda = 19{\cdot}62$. With this new value of λ the correction term '$0{\cdot}115w_0$' becomes '$0{\cdot}125w_0$', giving slightly altered values of $a^2\nabla^2 w_i$, shown in brackets in Fig. 9. Using these we find $\lambda = 19{\cdot}87$. Repeating this process again gives $\lambda = 19{\cdot}94$, which is 1·0 per cent. above the correct result. Fox's result is $\lambda = 19{\cdot}66$, which is 0·4 per cent. below the correct result, but since Rayleigh's Principle should over-estimate λ, his result is not necessarily the more accurate.

Acknowledgements

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ON THE FORCED VIBRATIONS OF QUASI-LINEAR SYSTEMS

By F. G. FRIEDLANDER
(*Dept. of Mathematics, The University, Manchester*)

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SUMMARY

In this paper we discuss the differential equation

$$\ddot{x}+\omega^2 x = kf(x, \dot{x}, t),$$

where ω, k are parameters, k being small (dots denote time-derivatives), and $f(x, \dot{x}, t)$ is periodic with period 2π in t. Using an approximate solution due to Kryloff and Bogoliuboff it is shown that, if $f(x, \dot{x}, t)$ is a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$, the 'amplitude' $(x^2+\dot{x}^2/\omega^2)^{\frac{1}{2}}$ settles down to an asymptotic range of variation of order k as $t \to \infty$ which depends on the initial conditions, unless ω differs only by order k from any one of a set of critical rational values. For these values, we shall say that 'subharmonic resonance' occurs. In this case the upper bound for the asymptotic range of variation of the amplitude is of order $k/|\omega-\omega_0|$, where ω_0 is the critical frequency nearest to ω. The extension of the argument to an equation in which $f(x, \dot{x}, t)$ is not a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$, but still periodic in t, is indicated. The asymptotic behaviour of the 'phase' $\tan^{-1}(x\omega/\dot{x})$ is also considered briefly.

The reduction of the case of 'subharmonic resonance' to an auxiliary differential equation of the first order is outlined, and it is shown that if either $\omega_0 = m/n$, where m and n are large relatively prime integers, or if $k/|\omega-\omega_0|$ is small, the results obtained in this way are in agreement with those obtained when subharmonic resonance does not occur.

1. NON-LINEAR differential equations of the type

$$\ddot{x}+\omega^2 x = kf(x, \dot{x}, t), \tag{1}$$

where dots indicate differentiation with respect to the time t, ω is a constant, k a small parameter, and $f(x, \dot{x}, t)$ a periodic function of t, say with period 2π, have been investigated extensively. They arise in the discussion of the forced vibrations of quasi-linear systems of one degree of freedom. In particular, two lines of attack have proved fruitful; the first is an application of a method due to Poincaré for the determination of periodic solutions (**1**); the second is a method of successive approximations developed by Kryloff and Bogoliuboff (**2**), which is closely related to other extensions of the method of 'variation of parameters' which have also been employed.

The object of this paper is to examine to what extent the properties of exact solutions of (1) can be deduced from the 'first approximation' of Kryloff and Bogoliuboff, to indicate how their theory can be extended to the case when $f(x, \dot{x}, t)$ is not a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$ (as

assumed by them), and to consider in what way their method fails when ω is near to certain critical values for which 'subharmonic resonance' occurs. It should be noted that an equation such as

$$\ddot{\xi}+\omega^2\xi+kF(\xi,\dot{\xi}) = E\sin t \tag{2}$$

(the more usual form of the equation for the forced vibrations of a quasi-linear system) can be reduced to the form (1) by putting

$$x = \xi+\frac{E}{\omega^2-1}\sin t, \tag{3}$$

and there is an analogous substitution if the right-hand side of (2) is any periodic function of t.

2. We begin by replacing (1) by two simultaneous equations of the first order. Let

$$x = r\sin\theta, \qquad \dot{x} = \omega r\cos\theta. \tag{4}$$

Then it is easily shown that

$$\dot{r} = \frac{1}{\omega}(\ddot{x}+\omega^2x)\cos\theta, \qquad \dot{\theta} = \omega-\frac{1}{\omega r}(\ddot{x}+\omega^2x)\sin\theta,$$

so that (1) is equivalent to

$$\left.\begin{aligned}\frac{dr}{dt} &= \frac{k}{\omega}f(r\sin\theta,\omega r\cos\theta,t)\cos\theta = kF(r,\theta,t)\\ \frac{d\theta}{dt} &= \omega-\frac{k}{\omega r}f(r\sin\theta,\omega r\cos\theta,t)\sin\theta = \omega+kG(r,\theta,t)\end{aligned}\right\}. \tag{5}$$

It may be noted at this stage that the considerations which will be put forward apply to any system of the form (5) provided that F and G are periodic functions of θ and t, with period 2π in both variables. It is obvious that $F(r,\theta,t)$ and $rG(r,\theta,t)$ satisfy the same conditions of continuity, boundedness, and differentiability with respect to r, θ, and t as $f(x,\dot{x},t)$ does with respect to x, $\dot{x}$, and t.

A question of some importance in applications is how any solution $r = r(t)$, $\theta = \theta(t)$ of (5), which takes given initial values r_0, θ_0 when $t = 0$, behaves asymptotically as $t\to\infty$. We shall be mainly concerned with this question. Let us first consider the behaviour of r, defined by

$$r = \left(x^2+\frac{\dot{x}^2}{\omega^2}\right)^{\frac{1}{2}}, \tag{4a}$$

which reduces to the amplitude of the simple harmonic vibration satisfying (1) when $k = 0$. A simple method of some generality for attacking this problem has been given elsewhere (**3**); here we shall follow Kryloff and Bogoliuboff, adopting a slightly different point of view.

3. Consider a function ρ defined by

$$\rho = r - ku(r, \theta, t), \tag{6}$$

where u is periodic in θ with period 2π and remains bounded as $t \to \infty$ for fixed r and θ. Then

$$\frac{d\rho}{dt} = k\left(F - \frac{\partial u}{\partial t} - \omega \frac{\partial u}{\partial \theta}\right) - k^2\left(F\frac{\partial u}{\partial r} + G\frac{\partial u}{\partial \theta}\right). \tag{7}$$

As the sign of $d\rho/dt$ is essentially given by the term of order k on the right-hand side, when k is sufficiently small, an obvious device is to choose u in such a way that this term assumes a suitable simple form. Now as

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2\pi T} \int_0^T \int_0^{2\pi} \left(\frac{\partial u}{\partial t} + \omega \frac{\partial u}{\partial \theta}\right) dt d\theta \\
&\quad = \lim_{T\to\infty} \frac{1}{2\pi T} \left\{ \int_0^{2\pi} [u(\theta, T) - u(\theta, 0)]\, d\theta + \omega \int_0^T [u(2\pi, t) - u(0, t)]\, dt \right\} \\
&\quad = \lim_{T\to\infty} \frac{1}{2\pi T} \int_0^{2\pi} [u(\theta, T) - u(\theta, 0)]\, d\theta = 0,
\end{aligned}$$

using the periodicity and boundedness of u, the mean value of the coefficient of k in (7) is equal to that of F, i.e. to

$$F_0(r) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F(r, \theta, t)\, d\theta dt = \frac{1}{4\pi^2 \omega} \int_0^{2\pi} \int_0^{2\pi} f(r \sin\theta, \omega r \cos\theta, t) \cos\theta\, d\theta dt. \tag{8}$$

We may therefore try to choose a function u for which this coefficient reduces to $F_0(r)$, or differs from $F_0(r)$ by less than any pre-assigned $\epsilon > 0$.

Let us assume, in the first place, that $f(x, \dot{x}, t)$ is a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$, so that $F(r, \theta, t)$ (and likewise $rG(r, \theta, t)$) is a trigonometric polynomial in θ and t whose coefficients are polynomials in r. Thus

$$F(r, \theta, t) = F_0(r) + \sum{}' F_{nm} e^{i(n\theta + mt)}, \tag{9}$$

where the summation extends over a finite set of pairs of integers (m, n), the pair $n = m = 0$ being excluded.

Then, if we can take

$$u = \sum{}' \frac{F_{nm}(r)}{i(n\omega + m)} e^{i(n\theta + mt)}, \tag{10}$$

the equation (7) becomes

$$\frac{d\rho}{dt} = kF_0(r) - k^2\left(F\frac{\partial u}{\partial r} + G\frac{\partial u}{\partial \theta}\right),$$

and this can be written in the form

$$\frac{d\rho}{dt} = kF_0(\rho)+k^2F_1(r,\theta,t,k,\omega), \tag{11}$$

where
$$F_1(r,\theta,t,k,\omega) = \frac{1}{k}[F_0(\rho)-F_0(r)]-F\frac{\partial u}{\partial r}-G\frac{\partial u}{\partial \theta} \tag{11a}$$

and F_1 remains continuous and bounded as $k \to 0$. Now this choice of u is obviously impossible unless, for all pairs of integers (n, m) which occur in the summation in (9),

$$n\omega+m \neq 0 \tag{12}$$

and we must at present assume that this holds. This condition is not satisfied, and (10) becomes meaningless when $\omega = -m/n$, where (n, m) is a pair of integers for which $F_{nm}(r)$ is not identically zero, and these 'critical' values of ω must be treated separately.

4. The advantage of choosing u in this manner is that the asymptotic behaviour of $\rho(t)$ can be deduced from (11) without difficulty, at least with an error of order k, and k is assumed to be small, Let

$$M = M(R', R'', k_0, \omega) = \max|F_1(r,\theta,t,k,\omega)|$$
$$(R' \leqslant r \leqslant R'',\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant t \leqslant 2\pi,\ 0 \leqslant k \leqslant k_0), \tag{13}$$

where $R' > 0$, and k_0 is some fixed (small) upper bound for k. (We cannot take $R' = 0$ as F_1 depends on G, and by (5) G tends to infinity when r tends to zero). We may note at once that $M \to \infty$ if ω approaches one of the 'critical' values just referred to. More exactly, if

$$\delta = \delta(\omega) = \min|n\omega+m| \tag{14}$$

where (n, m) runs through all the pairs of integers which occur in the summation in (9), then we can obviously take

$$M = \frac{C(R', R'', k_0)}{\delta(\omega)} = \frac{C}{\delta}. \tag{15}$$

If ω is kept fixed, and it is assumed that $\delta(\omega) > 0$, then this can be ignored for the present. Now it follows at once from (11) that $d\rho/dt$ and $F_0(\rho)$ have the same sign provided that

$$|F_0(\rho)| > kM. \tag{16}$$

To discuss the consequences of this result (which is the key to the asymptotic behaviour of $\rho(t)$ and hence also of $r(t)$) we shall begin by considering a simple, but typical, case. Suppose that $F_0(\rho)$ has a single positive zero $\rho = a$, such that

$$F_0(\rho) > 0 \quad (0 < \rho < a), \qquad F_0(\rho) < 0 \quad (\rho > a), \qquad F_0'(a) < 0. \tag{17}$$

Then the inequality (16) will hold in intervals (R', a') and (a'', R'') such that

$$a' < a < a'', \qquad F_0(a') = -F_0(a'') = kM \tag{18a}$$

and

$$a''-a' = O(k). \tag{18b}$$

Then clearly

$$\frac{d\rho}{dt} > 0 \quad (R' < \rho < a'), \qquad \frac{d\rho}{dt} < 0 \quad (a'' < \rho < R'').$$

Hence if, at any time $t = t_0$, $a' \leqslant \rho(t_0) \leqslant a''$, then also $a' \leqslant \rho(t) \leqslant a''$ for all $t > t_0$. For $\rho(t)$ can only leave the interval (a', a'') either by decreasing below a' or by increasing above a''; but as it is an increasing function of t whenever it is less than a', and a decreasing function of t whenever it exceeds a'', this is obviously impossible.

Again, if initially $\rho(0) = \rho_0$ is, for example, in (R', a'), then $\rho(t)$ must increase with t as long as $\rho < a'$; hence either $\rho(t) \to a'$ as $t \to \infty$, or, for some $t = t_0$, $\rho(t_0) = a'$, and in that case $a' \leqslant \rho \leqslant a''$ for all $t > t_0$, as has just been proved. Similarly if ρ_0 is in (a'', R'') then either $\rho(t) \to a''$ as $t \to \infty$, or $a' \leqslant \rho \leqslant a''$ from some finite $t = t_0$ and onwards. Thus asymptotically $\rho(t)$ varies only in the interval (a', a'') whose length is of order k. We may express this by saying that, given any $\epsilon > 0$ there is a t_0 such that

$$a'-\epsilon < \rho(t) < a''+\epsilon \quad (t > t_0) \tag{19}$$

(provided that $R' < \rho_0 < R''$) or, more concisely, by

$$a' \leqslant \lim_{t\to\infty} \rho(t) \leqslant \overline{\lim_{t\to\infty}}\, \rho(t) \leqslant a''. \tag{20}$$

In terms of r and θ this result becomes, by (6) and (10),

$$a'-\epsilon < r-k \sum{}' \frac{F_{nm}(r)}{i(n\omega+m)} e^{i(n\theta+mt)} < a''+\epsilon \quad (t > t_0). \tag{21}$$

For small k this result is obviously of considerable practical interest, as it gives the asymptotic range of amplitude variation with only a small error. It must be noted, however, that the t_0 in (19) is a function of k, and tends to infinity as $k \to 0$; this is to be expected, as for $k = 0$ the solution of (1) is

$$x = A \sin(\omega t+\alpha),$$

where A, α are arbitrary constants, and so $\rho(t) = A$ is constant and arbitrary, and in no way related to the zero $\rho = a$ of $F_0(\rho)$. Thus the smaller the departure from linearity, the greater the time required for the system to 'settle' to its asymptotic state.

We must now consider how this result depends on ω. From (15) it is obvious at once that (16) must be replaced by

$$|F_0(\rho)| > C\frac{k}{\delta(\omega)},$$

and hence the breadth of the interval of asymptotic variation, as determined by the method employed here, is given by

$$a''-a' = O\left(\frac{k}{\delta(\omega)}\right). \tag{22}$$

The result (21), or (20), is unaltered, but if it is to be of any practical use, $k/\delta(\omega)$ must be small. Hence, in particular, it is of little value if ω varies with k in such a way that

$$\omega = -\frac{m}{n}+\eta k, \tag{23}$$

where again (n, m) is a pair of integers for which $F_{nm}(r)$ is not identically zero, and η is a constant of order unity. For this type of 'critical range' of ω a different approach is obviously required.

5. In order to complete the discussion of our special example (17) we must also consider the choice of the lower limit R'. Since M depends linearly on G, and $G = O(1/r)$, it is clear that $M = O(1/R')$. Hence, if the inequality (16) is to hold for small values of r and so also of R', we require that

$$F_0(\rho) > \frac{kM_1}{R'}, \tag{24}$$

where M_1 is a constant, and by (6),

$$\rho \geqslant R'-kM_2 \tag{25}$$

where M_2 is the upper bound of $|u|$, u being defined by (10). If $F_0(0) \neq 0$, (24) is roughly equivalent to

$$R' > kM_1/F_0(0), \tag{26}$$

so that our conclusions concerning the asymptotic behaviour of ρ hold down to initial values of r of order k. If, as is more usually the case, $F_0(0) = 0$, some modification is required. Suppose, for example, that $F_0'(0) \neq 0$, so that (24) is approximately

$$F_0'(0)\rho > kM_1/R'.$$

Combining this with (25) it follows that we may take R' to be of order $k^{\frac{1}{2}}$ in this case; similarly, if $F_0(\rho) \sim \rho^\mu$ for small ρ, R' can be taken to be of order $k^{1/(1+\mu)}$. For initial values of r in the interval $(0, R')$ there may be solutions which remain small for all $t > 0$; these are unstable in the sense that if the system is disturbed it will in general drift away from this state of motion and settle down to a motion whose amplitude is in (a', a''). These unstable, permanently small, solutions, if periodic, have the period 2π of the driving force, i.e. they correspond to the 'fundamental' as distinct from the 'subharmonic' response of the system.

6. It is easy to extend the results obtained to the general case, when the polynomial $F_0(\rho)$ has several zeros. Suppose for instance that $F_0(\rho)$ has n positive simple zeros $\rho = a_1, a_2, \ldots, a_n$, so that $F_0(\rho)$ is alternately positive and negative in the intervals (a_i, a_{i+1}). Let a_i be a zero of $F_0(\rho)$ for which

$$F_0(\rho) > 0 \ (a_{i-1} < \rho < a_i), \ F_0(\rho) < 0 \ (a_i < \rho < a_{i+1}), \ F_0'(a_i) < 0. \quad (27)$$

For sufficiently small k, the inequalities

$$|F_0(\rho)| > kM, \qquad |F_0(\rho)| \leqslant kM$$

determine three intervals (a_{i-1}'', a_i'), (a_i', a_i''), (a_i'', a_{i+1}') such that

$$\left.\begin{aligned} F_0(a_{i-1}'') = F_0(a_i') = -F_0(a_i'') = -F_0(a_{i+1}') = kM \\ |F_0(\rho)| > kM \quad (a_{i-1}'' < \rho < a_i', \quad a_i'' < \rho < a_{i+1}') \\ |F(\rho)| \leqslant kM \quad (a_i' \leqslant \rho \leqslant a_i'') \\ a_{i-1}''-a_{i-1} = O(k), \quad a_i''-a_i' = O(k), \quad a_{i+1}-a_{i+1}' = O(k) \end{aligned}\right\}. \quad (28)$$

It then follows, as in the special case which has just been discussed, that if

$$a_{i-1}'' < \rho_0 < a_{i+1}' \quad (29\,a)$$

then

$$a_i' \leqslant \varliminf_{t\to\infty} \rho(t) \leqslant \varlimsup_{t\to\infty} \rho(t) \leqslant a_i''. \quad (29\,b)$$

Thus the initial values of (r, θ) can be classified. If we interpret (r, θ) as polar coordinates in a plane, then the curves

$$r-ku(r, \theta, 0) = \rho$$

are a congruence of closed curves. To every zero a_i for which (27) holds corresponds a domain $\mathscr{A}_i$ defined by

$$a_{i-1}'' < r-ku(r, \theta, 0) < a_{i+1}', \quad (30\,a)$$

with the property that the asymptotic values of a solution whose initial values lie in $\mathscr{A}_i$ are in the closed domain $\mathscr{D}_i$ defined by

$$a_i' \leqslant r-ku(r, \theta, t) \leqslant a_i'', \quad (30\,b)$$

and this is an annulus with width of order k, so that we may also write

$$\varlimsup_{t\to\infty} |r-ku(r, \theta, t)-a_i| = O(k). \quad (31)$$

The solutions whose initial values lie in $\mathscr{D}_i$ possess a type of stability analogous to 'orbital stability' which may, perhaps, be called 'amplitude stability'. In the case envisaged here, the domains $\mathscr{A}_i$ are separated by annular domains of width $O(k)$ which are completely unstable in the sense that, though certain solutions may remain in them for all t, if such a solution is disturbed sufficiently (though only to order k) it drifts away from the domain and settles down in one of the 'stable' annuli $\mathscr{D}_i$. The

neighbourhood of the origin must be treated separately. The 'unstable' case has already been discussed, and the extension to the 'stable' case is obvious.

7. A similar argument can be developed for the coordinate θ. Let

$$\phi = \theta - kv(r, \theta, t) \tag{32}$$

and suppose that rG, like F, is a trigonometric polynomial in θ and t whose coefficients are polynomials in r. Then

$$G = G_0(r) + \sum{}' G_{nm}(r) e^{i(n\theta+mt)}. \tag{33}$$

If we assume once more that $m+n\omega \neq 0$ for all pairs (n, m) which occur in this summation, then we may put

$$V(r, \theta, t) = \sum{}' \frac{G_{nm}(r)}{i(n\omega+m)} e^{i(n\theta+mt)} \tag{34}$$

and obtain

$$\frac{d\phi}{dt} = \omega + kG_0(r) + O(k^2),$$

or

$$\frac{d\phi}{dt} = \omega + kG_0(\rho) + O(k^2). \tag{35}$$

Now as, if we consider solutions whose initial values lie in $\mathscr{A}_i$, $|\rho - a_i|$ is asymptotically of order k, it follows that

$$\overline{\lim_{t\to\infty}} \left| \frac{\phi - \phi_0}{t} - \omega - kG(a_i) \right| = O(k^2). \tag{36}$$

If, in particular, the initial values lie in $\mathscr{D}_i$, this can be replaced by

$$\left| \frac{\phi - \phi_0}{t} - \omega - kG(a_i) \right| = O(k^2). \tag{37}$$

From this inequality, a necessary condition for the existence of subharmonic solutions of the original equation (1) can be derived. Such a solution is by definition periodic with least period $2n\pi$, where n is an integer greater than one (any periodic solution of (1) has necessarily a period which is 2π or an integral multiple of 2π). The corresponding $\rho(t)$ also has period $2n\pi$, while $\phi(t)$ is increased by $2m\pi$ if t is increased by $2n\pi$, m being another integer. Hence, applying (37) with $t = 2n\pi$, it follows that

$$\omega + kG(a_i) + O(k^2) = \frac{m}{n}, \tag{38}$$

so that subharmonics can only exist for given m and n and arbitrarily small k, if ω differs from the rational number m/n by $O(k)$. (In particular, this includes the 'critical' ranges of ω for which our present method breaks down.) Alternatively, n may be of order $1/k$ or larger.

If no subharmonics exist, one would expect that among the solutions whose initial values lie in $\mathscr{D}_i$ there are almost periodic 'drifts' of the form

$$x = \{a_i + kR(t, \alpha+\sigma t)\}\sin\{\sigma t+\alpha+kS(t,\alpha+\sigma t)\}, \tag{39}$$

where R, S have period 2π in both arguments, σ is an irrational number depending on a_i, and α is an arbitrary constant (**4**). Now if ρ and ϕ are introduced as dependent variables, then it is found that they satisfy differential equations of the form

$$\left.\begin{aligned} \frac{d\rho}{dt} &= kF_0(\rho)+k^2F_2(\rho,\phi,t,k) \\ \frac{d\phi}{dt} &= \omega+kG_0(\rho)+k^2G_2(\rho,\phi,t,k) \end{aligned}\right\} \tag{40}$$

where F_2, G_2 are periodic in both ϕ and t with period 2π, and remain continuous and bounded as $k \to 0$. Starting with these equations, it can be shown that every solution whose initial values lie in $\mathscr{A}_i$ tends asymptotically to a solution of the form (39) or to a subharmonic solution; the proof, which is lengthy and depends on ideas which have not been introduced in this paper, will be omitted.† It may be added that if the terms of order k^2 are omitted in (40), the equations of the 'first crude approximation' of Kryloff and Bogoliuboff (**2**) are obtained, to whom the equations (40) are also due. These approximate equations give an estimate of the manner in which an arbitrary solution approaches its asymptotic form, and many examples of their application will be found in reference (**2**).

8. If the function $f(x, \dot{x}, t)$ is not a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$, the method developed here seems to break down. For then the finite sum in (9) must be replaced by a Fourier series, and even if ω is irrational, so that (12) is satisfied for all pairs of integers (n, m), the convergence of (10) is not a necessary consequence of that of (9), and its discussion is a matter of some difficulty. But, by an extension of Weierstrass's theorem, we can with any preassigned $\epsilon > 0$, find an $F^*(r, \theta, t)$ which is a trigonometric polynomial in θ and t with coefficients which are polynomials in r such that

$$|F(r,\theta,t)-F^*(r,\theta,t)| < \epsilon \quad (R' < r < R''), \tag{41}$$

the order of the trigonometric polynomial F^* tending to infinity as $\epsilon \to 0$. Now if we define ρ^* by

$$\rho^* = r-ku^*, \tag{42}$$

where u^* is the trigonometric polynomial satisfying

$$\frac{\partial u^*}{\partial t}+\omega\frac{\partial u^*}{\partial\theta} = F^*-F_0^*(r) \tag{43}$$

† A result of this type seems to have been proved by Kryloff and Bogoliuboff (**5**), presumably with $f(x, \dot{x}, t)$ a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$. A variant of this proof, derived independently, in which this restrictive assumption is dispensed with, will, it is hoped, be published elsewhere.

(where $F_0^*(r)$ is the mean value of F^*), it follows that

$$\frac{d\rho^*}{dt} = kF_0^*(r)+k(F-F^*)-k^2\left(F\frac{\partial u^*}{\partial r}+G\frac{\partial u^*}{\partial \theta}\right). \tag{44}$$

We can now repeat our arguments with F_0^* instead of F_0, and it is found that results such as (20) or (21) hold except that

$$a''-a' = O\left(\frac{k}{\delta^*}+\epsilon\right), \tag{45}$$

where δ^* is defined in relation to F^* as δ was defined for F. Thus δ^* depends on the order of the polynomial F^*, and in general it decreases steadily to zero as this order tends to infinity, which happens if we let $\epsilon \to 0$. The set of 'critical values' also depends on ϵ, and becomes dense in $(0, \infty)$ as $\epsilon \to 0$ unless F is of a special form. Hence we cannot let $\epsilon \to 0$; but there will be a value of ϵ for which $\epsilon+k/\delta^*$ is as small as possible, and this value will give the best possible bounds. It is not easy to discuss the details of this process in general, but the character of the results obtained here persists, though with a certain loss of precision. (This result may be compared with that obtained by a different method elsewhere (**3**).)

9. Returning to the simpler case of an $f(x, \dot{x}, t)$ which is a polynomial in x, $\dot{x}$, $\cos t$, and $\sin t$, we must now consider the 'critical' ranges of ω. For this purpose, let us now assume that

$$\omega = \frac{q}{p}+\eta k, \tag{46}$$

where p, q are integers and η is a parameter of order unity. This case can be dealt with by using the methods originated by van der Pol and by Appleton, and developed by Cartwright and Littlewood (**6**, **7**) for the triode oscillator synchronization problem. We shall consider it briefly, and show that if either p and q are large, or if η is large, the results obtained will approximate to those which have just been found from the first approximation of Kryloff and Bogoliuboff.

Our equations (5) now become

$$\left.\begin{aligned}\frac{dr}{dt} &= k\{F_0(r)+\sum{}' F_{nm}(r)e^{i(n\theta+mt)}\}\\ \frac{d\theta}{dt} &= (q/p)+k\{\eta+G_0(r)+\sum{}' G_{nm}(r)e^{i(n\theta+mt)}\}\end{aligned}\right\} \tag{47}$$

and these can be considered as defining the motion of a point in a plane whose polar coordinates are (r, θ) (the phase plane of the equations). Since the right-hand sides of these equations have period of 2π in t, it is

sufficient to know the trajectories starting at initial values (r_0, θ_0) in the time interval $(0, 2\pi)$ in order to deduce the trajectories in the complete semi-infinite time interval $(0, \infty)$. For by the uniqueness theorem relating to the solutions of differential equations, if the trajectories are given by

$$r = R(r_0, \theta_0, t), \quad \theta = \Theta(r_0, \theta_0, t) \qquad (0 \leqslant t \leqslant 2\pi), \tag{48}$$

then

$$r = R(r_1, \theta_1, t-2\pi), \quad \theta = \Theta(r_1, \theta_1, t-2\pi) \qquad (2\pi < t \leqslant 4\pi)$$

and generally

$$r = R(r_s, \theta_s, t-2s\pi), \quad \theta = \Theta(r_s, \theta_s, t-2s\pi) \qquad (2s\pi < t \leqslant 2(s+1)\pi),$$

where

$$r_s = r(2s\pi) = R(r_{s-1}, \theta_{s-1}, 2\pi), \qquad \theta_s = \theta(2s\pi) = \Theta(r_{s-1}, \theta_{s-1}, 2\pi).$$

Instead of using the interval $(0, 2\pi)$ in this way, we can use the interval $(0, 2p\pi)$. The trajectories will therefore be completely determined if we can find the points (r_{sp}, θ_{sp}) $(s = 1, 2, \ldots)$ corresponding to the point (r_0, θ_0). Now

$$r_{(s+1)p} = R(r_{sp}, \theta_{sp}, 2\pi p), \qquad \theta_{(s+1)p} = \Theta(r_{sp}, \theta_{sp}, 2\pi p). \tag{49}$$

We are therefore led to study the mapping of the phase plane into itself defined by

$$r' = R(r, s, 2p\pi), \qquad \theta' = \Theta(r, s, 2p\pi). \tag{50}$$

Moreover, we can subtract any integral multiple of 2π from θ', as (r, θ) and $(r, \theta+2\pi)$ define the same point. Now it follows from (47) that

$$r_p = r_0 + k \int_0^{2p\pi} F\left(r_0, \theta_0 + \frac{q}{p}t, t\right) dt + O(k^2p^2),$$

$$\theta_p = \theta_0 + 2q\pi + k\left\{2p\pi\eta + \int_0^{2p\pi} G\left(r_0, \theta_0 + \frac{q}{p}t, t\right) dt\right\} + O(k^2p^2).$$

Omitting the term $2q\pi$ on the right-hand side of the last equation, and using the expansions of F, G it follows that

$$\left.\begin{aligned} r' &= r + 2\pi pk\left\{F_0(r) + \sum_l F_{lp,-lq}(r)e^{ilp\theta}\right\} + O(k^2p^2) \\ \theta' &= \theta + 2\pi pk\left\{\eta + G_0(r) + \sum_l G_{lp,-lq}(r)e^{ilp\theta}\right\} + O(k^2p^2) \end{aligned}\right\}. \tag{51}$$

10. To discuss the mapping (51) for small k, we need the result that, if ρ and ψ are the solutions of the differential equations

$$\left.\begin{aligned} \frac{d\rho}{dt} &= k\left\{F_0(\rho) + \sum_l F_{lp,-lq}(\rho)e^{ilp\psi}\right\} \\ \frac{d\psi}{dt} &= k\left\{\eta + G_0(\rho) + \sum_l G_{lp,-lq}(\rho)e^{ilp\psi}\right\} \end{aligned}\right\} \tag{52}$$

then
$$\left.\begin{aligned} r(2sp\pi)-\rho(2sp\pi) &= O(kp)\\ [\theta(2sp\pi)-\psi(2sp\pi)]\ (\mathrm{mod}\ 2\pi) &= O(kp)\end{aligned}\right\}. \tag{53}$$

The proof of this theorem will be omitted here. Then we can determine the points (r_{sp}, θ_{sp}), and hence the complete trajectories, with an error of order k, if we can solve the equations (52). This is a simpler problem than the original one, as the right-hand sides of (52) do not depend on t. The asymptotic behaviour of the solutions of (52) is fairly simple. They either tend to a 'critical point', i.e. to a point at which both $d\rho/dt$ and $d\psi/dt$ vanish, or they approach a closed trajectory asymptotically. The first case corresponds to a subharmonic of order p of the original equation (7), while the second one corresponds to a 'drift' solution of the type (39) (this result is due to Cartwright and Littlewood, but unpublished). Obviously each case must be considered on its merits, but there are two generally valid results which must be mentioned.

In the first place, if we suppose that the series expansions in (47) are not finite, but infinite double Fourier series, then the F_{nm}, G_{nm} tend to zero as $|n|+|m| \to \infty$, by the Riemann–Lebesgue lemma. Hence if both p and q are large, and pk small, all the terms in (52) except F_0 and $\eta+G_0$ will be negligible, and these equations reduce approximately to a form satisfied by the first approximation of Kryloff and Bogoliuboff (with $\psi = \phi+\omega t$).

Secondly, if η is large, but $k\eta$ still small, and if we consider the single first-order equation equivalent to (52),

$$\frac{d\rho}{d\psi} = \frac{F_0(\rho)+\sum F_{lp,-lq}(\rho)\exp(ilp\psi)}{\eta+G_0(\rho)+\sum G_{lp,-lq}(\rho)\exp(ilp\psi)}, \tag{54}$$

then (for sufficiently large η) the only critical point will be the origin, which corresponds to a fixed point of the mapping

$$r' = R(r, \theta, 2\pi), \qquad \theta' = \Theta(r, \theta, 2\pi)$$

and hence to a solution of the original equation of period 2π. Also, there will be closed trajectories within $O(1/\eta)$ of the circles

$$\rho = a_i, \qquad F_0(a_i) = 0. \tag{55}$$

This is in complete agreement with the results already obtained in the non-critical case. To prove it, we write (54) in the form

$$\frac{d\rho}{d\psi} = \frac{1}{\eta}\{F_0(\rho)+\sum F_{lp,-lq}(\rho)e^{ilp\psi}\}+O\left(\frac{1}{\eta^2}\right). \tag{56}$$

We are assuming that η is so large that the denominator of the right-hand side of (54), which is proportional to the right-hand side of the second equation of (52) is positive, so that the trajectories encircle the origin in

the clockwise sense. Thus if $\rho = \rho_0$ for $\psi = 0$, then $\rho' = \rho(2\pi)$, where ρ is considered as a function of ψ, is given by

$$\rho' = \rho_0 + \frac{1}{\eta}\int_0^{2\pi} \{F_0(\rho) + \sum F_{lp,-lq}(\rho_0) e^{ilp\psi}\}\, d\psi + O\left(\frac{1}{\eta^2}\right)$$

or

$$\rho' = \rho_0 + \frac{2\pi}{\eta} F_0(\rho_0) + O\left(\frac{1}{\eta^2}\right). \tag{57}$$

Hence the closed trajectories of (56), for which $\rho' = \rho_0$, are within $O(1/\eta)$ of the zeros of $F_0(\rho)$. Moreover, it can be shown from (57) that a closed trajectory associated with a zero $\rho = a$ of $F_0(\rho)$ is stable if $F_0'(a) < 0$, and completely unstable if $F_0'(0) > 0$ (if $F_0'(a) = 0$ the trajectory may be semi-stable, and in any case the interpretation in terms of the exact solution of the mapping equations (51) becomes difficult). The corresponding exact solutions of the (r, θ)-equations are orbitally stable, or completely unstable 'drifts', respectively. Thus we have, for a stable drift associated with a zero $\rho = a$ of $F_0(\rho)$, that

$$|r - a_i| = O(k) + O\left(\frac{1}{\eta}\right) = O\left(k + \frac{k}{\delta}\right) = O\left(\frac{k}{\delta}\right),$$

where

$$\delta = n\omega - m$$

and this is obviously in agreement with our earlier results.

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A NOTE ON THE DIFFRACTION OF A CYLINDRICAL WAVE BY A PERFECTLY CONDUCTING HALF-PLANE

By P. C. CLEMMOW

(*Electrical Engineering Department, Imperial College, London*)

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SUMMARY

The most easily derived solutions in diffraction theory are for the case of incident plane waves. The solution for a line-source can then be obtained by a further integration, but this method, although well known in some fields, does not seem to have been applied in diffraction theory. Because of its importance in more complicated problems it is here illustrated by the example of a line-source in the presence of an infinitely thin, perfectly conducting half-plane. The introduction of Hankel functions is avoided, and the solution appears in a useful form, equivalent to that given by Macdonald and analogous to Sommerfeld's famous solution for an incident plane wave.

1. Introduction

THE problem to be considered is the two-dimensional one illustrated in Fig. 1; T is a line-source, situated at (r_0, θ_0), propagating the cylindrical wave (omitting the time factor)

$$\sqrt{(\tfrac{1}{2}\pi)}e^{-i\pi/4}H_0^{(2)}(kR) \sim \frac{e^{-ikR}}{\sqrt{(kR)}}; \tag{1}$$

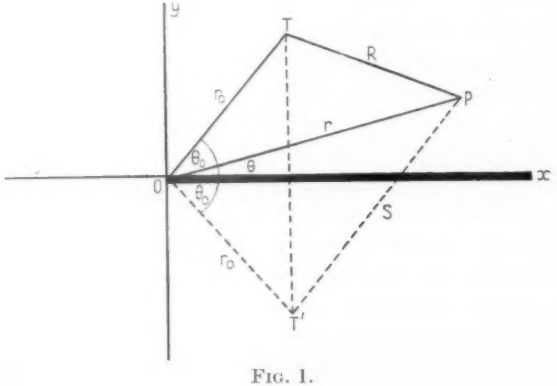

FIG. 1.

and the half-plane $y = 0, x > 0$ is occupied by an infinitely thin, perfectly conducting sheet. The problem is essentially scalar, being expressed in terms of E_z or H_z according to the polarization. For convenience, attention will be confined to the case where $\mathbf{H} = (0, 0, H_z)$.

Carslaw (**1**, **2**) gave the solution in a form equivalent to

$$H_z = u(\theta_0)+u(-\theta_0), \tag{2}$$

where

$$u(\theta_0) = \frac{e^{i\pi/4}}{4\sqrt{(2\pi)}} \int \frac{\sin \tfrac{1}{2}\alpha}{\cos \tfrac{1}{2}\alpha - \cos \tfrac{1}{2}(\theta-\theta_0)} H_0^{(2)}[k\sqrt{\{r^2+r_0^2-2rr_0\cos(\alpha-\theta_0)\}}]\, d\alpha, \tag{3}$$

the integration being over a path in the upper half of the complex plane, going from $\pi+i\infty$ under the branch-point to $-\pi+i\infty$.

In his first paper Carslaw (**1**) prefaced the derivation of his solution with the words '. . . the results . . . are not obtained in a workable form owing to the necessity for introducing Bessel's functions into the integrals . . .'. But later Macdonald (**3**) showed that (2) could be usefully transformed into

$$H_z = \frac{e^{i\pi/4}}{\sqrt{(2\pi)}} \int_{-\infty}^{\xi_0} e^{-ikR\,\mathrm{ch}\,\xi}\, d\xi + \frac{e^{i\pi/4}}{\sqrt{(2\pi)}} \int_{-\infty}^{\xi_1} e^{-ikS\,\mathrm{ch}\,\xi}\, d\xi, \tag{4}$$

where $R\,\mathrm{sh}\,\xi_0 = 2\sqrt{(r_0 r)}\cos \tfrac{1}{2}(\theta-\theta_0)$, $S\,\mathrm{sh}\,\xi_1 = 2\sqrt{(r_0 r)}\cos \tfrac{1}{2}(\theta+\theta_0)$, and sh and ch denote the hyperbolic sine and cosine respectively.

In this note the problem is treated by a different method which does not involve Hankel functions. Since the final result is a form essentially equivalent to (4), the reason for undertaking the analysis must be briefly stated.

The use of integral equations has recently led to the rigorous solution of several new problems in diffraction theory, one of which has important applications in radio propagation (**4**). Because the physical discontinuities are planar, the fundamentally easiest problems are those for which the incident wave is plane: in the radio propagation problem, however, it is essential to deal with a transmitter which is conveniently taken as a line-source. Now, although the dictum that general solutions can be built up from plane-wave solutions is perhaps more often preached than practised, the procedure has proved extremely useful in propagation theory, in particular in connexion with the problem just mentioned. This latter problem is a rather complicated generalization of the present one; but it seems desirable that the results to be given in this note should be set out independently, both because of their interest in a different field and because it is best to illustrate the technique by applying it to a relatively

simple example. Furthermore, the solution (4) is perhaps not as well known as it deserves to be, although it is quoted by Bateman (5). Baker and Copson (6), for instance, only mention Carslaw's form (2), and indeed the author was not aware of Macdonald's result until after it had been obtained independently by the present method. It will be seen later that (4) is virtually as simple as Sommerfeld's famous solution for an incident plane wave.

2. The method

First, consider the problem when the incident field is the plane wave

$$H_z^i = e^{ikr\cos(\theta-\alpha)} \quad (0 < \alpha < \pi). \tag{5}$$

The total field may be conveniently written as

$$H_z^p = H_z^{pG} + H_z^{pD}, \tag{6}$$

where the affix p indicates that the incident wave is plane, and the affixes G and D imply 'geometrical optics' and 'diffraction' respectively. Then Sommerfeld's solution is equivalent to

$$H_z^{pD} = \begin{cases} \dfrac{i}{\pi}\displaystyle\int_{S(\theta)} \frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{\cos\beta+\cos\alpha} e^{-ikr\cos(\theta-\beta)}\,d\beta, & \text{for} \quad y > 0, \\ -\dfrac{i}{\pi}\displaystyle\int_{S(2\pi-\theta)} \frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{\cos\beta+\cos\alpha} e^{-ikr\cos(\theta+\beta)}\,d\beta, & \text{for} \quad y < 0, \end{cases} \tag{7}$$

where $S(\theta)$ denotes the path of 'steepest descents' passing through the real angle θ, as in Fig. 2.

Now consider the line-source at (r_0, θ_0) where $0 < \theta_0 < \pi$. Introducing the appropriate phase factor $\exp[-ikr_0\cos(\theta_0-\alpha)]$, Sommerfeld's integral representation of the Hankel function may be written

$$H_0^{(2)}(kR) = \frac{1}{\pi}\int_{S(\frac{1}{2}\pi)} e^{-ikr_0\cos(\theta_0-\alpha)} e^{ikr\cos(\theta-\alpha)}\,d\alpha \quad (y < r_0\sin\theta_0). \tag{8}$$

The form of (8) indicates that the solution for an incident field (1) can be obtained from that for an incident field (5) on multiplying by the factor

$$(2\pi)^{-\frac{1}{2}} e^{-i\pi/4} e^{-ikr_0\cos(\theta_0-\alpha)} \tag{9}$$

and integrating with respect to α over the path $S(\frac{1}{2}\pi)$.

A point to be noticed is that α is complex on $S(\frac{1}{2}\pi)$, whereas it is normally

taken to be real. But this introduces no difficulty, for (6) and (7) are still valid if it is appreciated that, for $0 < \operatorname{re}\alpha < \pi$,

$$H_z^{pG} = \begin{cases} e^{ikr\cos(\theta-\alpha)} + e^{ikr\cos(\theta+\alpha)} & \text{for } \alpha \text{ to the left of } S(\pi-\theta), \\ e^{ikr\cos(\theta-\alpha)} & \text{for } \alpha \text{ to the right of } S(\pi-\theta) \\ & \qquad \text{and } S(\theta-\pi), \\ 0 & \text{for } \alpha \text{ to the right of } S(\pi-\theta) \\ & \qquad \text{and to the left of } S(\theta-\pi), \end{cases} \tag{10}$$

relations which yield the usual 'geometrical optics' field when α is real.

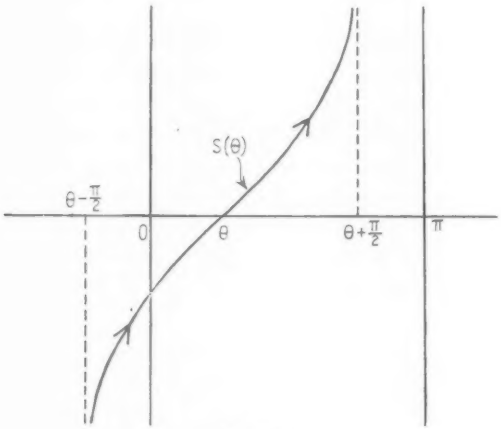

FIG. 2.

For a fixed θ, therefore, the terms in (6) are continuous for all α on $S(\frac{1}{2}\pi)$, and the suggested procedure leads to the solution of our problem.

3. The solution

The solution that we have to consider is

$$H_z = \frac{e^{-i\pi/4}}{\sqrt{(2\pi)}} \int_{S(\frac{1}{2}\pi)} H_z^{pG} e^{-ikr_0\cos(\theta_0-\alpha)}\, d\alpha + \frac{e^{-i\pi/4}}{\sqrt{(2\pi)}} \int_{S(\frac{1}{2}\pi)} H_z^{pD} e^{-ikr_0\cos(\theta_0-\alpha)}\, d\alpha, \tag{11}$$

where H_z^{pG} is given by (10) and H_z^{pD} by (7). To separate this field into 'geometrical optics' and 'diffraction' terms, the path of integration for α in the second expression in (11) must be taken through the 'predominant' angle θ_0. The poles of the integrand of (7), regarded as a function of α, are given by $\cos\alpha = -\cos\beta$, where β lies on $S(\theta)$ or $S(2\pi-\theta)$ according as to y is positive or negative: in transposing the α-path of integration from $S(\frac{1}{2}\pi)$ to $S(\theta_0)$ these poles may be crossed, and their residues must then

be allowed for. The contribution of these residues combines with the first expression in (11) to yield the 'geometrical optics' term of the solution. In fact, if we write (11) as

$$H_z = H_z^G + H_z^D, \tag{12}$$

it is easy to check, from a consideration of the different cases, that the correct form of H_z^G is obtained, namely

$$H_z^G = \begin{cases} \sqrt{(\frac{1}{2}\pi)}e^{-i\pi/4}[H_0^{(2)}(kR)+H_0^{(2)}(kS)] & \text{for} \quad 0 < \theta < \pi-\theta_0, \\ \sqrt{(\frac{1}{2}\pi)}e^{-i\pi/4}H_0^{(2)}(kR) & \text{for} \quad \pi-\theta_0 < \theta < \pi+\theta_0, \\ 0 & \text{for} \quad \pi+\theta_0 < \theta < 2\pi; \end{cases} \tag{13}$$

and then, for all values of y, the diffraction term may be written

$$H_z^D = \frac{e^{i\pi/4}}{\pi\sqrt{(2\pi)}} \int_{S(0)} \int_{S(0)} \frac{\cos\frac{1}{2}(\alpha+\theta_0)\cos\frac{1}{2}(\beta+\theta)}{\cos(\alpha+\theta_0)+\cos(\beta+\theta)} e^{-ik(r_0\cos\alpha+r\cos\beta)}\, d\alpha d\beta, \tag{14}$$

the symmetry in r_0, θ_0 and r, θ being demanded by the reciprocity theorem.

Equations (13) and (14) give one form of the solution. It is convenient to express (14) as

$$H_z^D = I(\theta_0)+I(-\theta_0), \tag{15}$$

where

$$I(\theta_0) = \frac{e^{i\pi/4}}{4\pi\sqrt{(2\pi)}} \int_{S(0)} \int_{S(0)} \frac{e^{-ik(r_0\cos\alpha+r\cos\beta)}}{\cos\frac{1}{2}(\alpha+\beta+\theta_0+\theta)}\, d\alpha d\beta. \tag{16}$$

By taking $\alpha+\beta$ as a new variable it may be seen that this can lead to Carslaw's form (2).

4. The reduction of the solution

Multiplying top and bottom of the integrand in (16) by

$$4\cos\tfrac{1}{2}(\alpha-\beta+\theta_0+\theta)$$

and discarding that part which is an odd function of β, we have

$$I(\theta_0) = \frac{e^{i\pi/4}}{8\pi\sqrt{(2\pi)}} \int_{S(0)} \int_{S(0)} \left[\frac{1}{\cos\frac{1}{2}(\alpha+\theta_0+\theta)-\sin\frac{1}{2}\beta} + \right.$$

$$\left. + \frac{1}{\cos\frac{1}{2}(\alpha+\theta_0+\theta)+\sin\frac{1}{2}\beta}\right] \cos(\tfrac{1}{2}\beta) e^{-ik(r_0\cos\alpha+r\cos\beta)}\, d\alpha d\beta. \tag{17}$$

In the second of these integrals change α to $-\alpha$ and then recombine the two. This gives

$$I(\theta_0) = \frac{e^{i\pi/4}}{2\pi\sqrt{(2\pi)}} \times$$

$$\times \int_{S(0)} \int_{S(0)} \frac{\cos\frac{1}{2}(\theta_0+\theta)\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\, e^{-ik(r_0\cos\alpha+r\cos\beta)}\, d\alpha d\beta}{(\cos\alpha-1)+(\cos\beta-1)-4\sin\frac{1}{2}(\theta_0+\theta)\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta+2\cos^2\frac{1}{2}(\theta_0+\theta)}. \tag{18}$$

Now make the 'steepest descents' substitutions

$$x = \sqrt{2}\, e^{-i\pi/4} \sin \tfrac{1}{2}\alpha, \qquad y = \sqrt{2}\, e^{-i\pi/4} \sin \tfrac{1}{2}\beta, \tag{19}$$

so that

$$I(\theta_0) = -\frac{e^{i\pi/4}}{\pi\sqrt{(2\pi)}} e^{-ik(r_0+r)} \cos \tfrac{1}{2}(\theta_0+\theta) \times$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-k(r_0 x^2+ry^2)}}{x^2+y^2+2\sin\tfrac{1}{2}(\theta_0+\theta)xy+2i\cos^2\tfrac{1}{2}(\theta_0+\theta)}\, dxdy. \tag{20}$$

Let $R_1 = r_0+r$, and make the polar substitutions

$$x = \sqrt{(R_1/r_0)}\rho\cos\phi, \qquad y = \sqrt{(R_1/r_0)}\rho\sin\phi. \tag{21}$$

Then we may write

$$I(\theta_0) = -\frac{e^{i\pi/4}}{\pi\sqrt{(2\pi)}} e^{-ikR_1} \cos\tfrac{1}{2}(\theta_0+\theta) \int_0^{\infty} \rho J(\rho) e^{-kR_1\rho^2}\, d\rho, \tag{22}$$

where

$$J(\rho) = \int_0^{2\pi} \frac{d\phi}{\rho^2[\sqrt{(r/r_0)}\cos^2\phi+\sqrt{(r_0/r)}\sin^2\phi+2\sin\tfrac{1}{2}(\theta_0+\theta)\sin\phi\cos\phi]+ \\ +\{2i\sqrt{(r_0 r)}/R_1\}\cos^2\tfrac{1}{2}(\theta_0+\theta)}; \tag{23}$$

$J(\rho)$ can be evaluated by the standard technique of putting $z = \exp(i\phi)$. The path of integration then becomes the unit circle, and it is only necessary to calculate the residues of the enclosed poles. Without going into details, it will be found that there are four poles altogether, given by $z = \pm p_1$, $z = \pm p_2$, say; but $|p_1 p_2| = 1$, so that only two poles lie within the unit circle. The result of the integration is

$$J(\rho) = 2\pi|\sec\tfrac{1}{2}(\theta_0+\theta)|\left[\rho^4+2i\rho^2-\frac{4r_0 r_1}{R_1^2}\cos^2\tfrac{1}{2}(\theta_0+\theta)\right]^{-\frac{1}{2}}, \tag{24}$$

where the square root is the branch in the first quadrant of the complex plane.

Referring to the geometry of Fig. 1, (22) evidently becomes

$$I(\theta_0) = \mp\sqrt{\frac{2}{\pi}} e^{i\pi/4} e^{-ikR_1} \int_0^{\infty} \frac{\rho e^{-kR_1\rho^2}}{\sqrt{\{[\rho^2+i((R_1-S)/R_1)][\rho^2+i((R_1+S)/R_1)]\}}}\, d\rho,$$
$$\mp \text{ for } \cos\tfrac{1}{2}(\theta_0+\theta) \gtrless 0. \tag{25}$$

The substitution

$$\rho^2+i\left(\frac{R_1-S}{R_1}\right) = i\frac{S}{R_1}\lambda^2$$

then gives

$$I(\theta_0) = \mp\sqrt{\frac{2}{\pi}} e^{i\pi/4} e^{-ikS} \int_{\sqrt{\{(R_1-S)/S\}}}^{\infty} \frac{e^{-ikS\lambda^2}}{\sqrt{(\lambda^2+2)}}\, d\lambda, \quad \mp \text{ for } \cos\tfrac{1}{2}(\theta_0+\theta) \gtrless 0. \tag{26}$$

The corresponding expression for $I(-\theta_0)$ will be

$$I(-\theta_0) = \mp\sqrt{\frac{2}{\pi}}e^{i\pi/4}e^{-ikR}\int_{\sqrt{\{(R_1-R)/R\}}}^{\infty}\frac{e^{-ikR\lambda^2}}{\sqrt{(\lambda^2+2)}}\,d\lambda, \quad \mp \text{ for } \cos\tfrac{1}{2}(\theta_0-\theta)\gtrless 0. \tag{27}$$

Now we can write the incident field (1) in the form

$$\sqrt{\frac{2}{\pi}}e^{i\pi/4}e^{-ikR}\int_{-\infty}^{\infty}\frac{e^{-ikR\lambda^2}}{\sqrt{(\lambda^2+2)}}\,d\lambda; \tag{28}$$

and the 'reflected' field is given by the same expression with S replacing R. Using (28), the diffraction term (15), determined by (26) and (27), can be neatly combined with the geometrical optics term (13). In order not to have to specify changes in the signs of the square roots it is convenient to write

$$\left.\begin{aligned}\epsilon_1 &= 2\sqrt{\left(\frac{kr_0r}{R_1+R}\right)}\cos\tfrac{1}{2}(\theta_0-\theta) = \pm\sqrt{\{k(R_1-R)\}}, \quad \pm \text{ for } \cos\tfrac{1}{2}(\theta_0-\theta)\gtrless 0,\\ \epsilon_2 &= 2\sqrt{\left(\frac{kr_0r}{R_1+S}\right)}\cos\tfrac{1}{2}(\theta_0+\theta) = \pm\sqrt{\{k(R_1-S)\}}, \quad \pm \text{ for } \cos\tfrac{1}{2}(\theta_0+\theta)\gtrless 0.\end{aligned}\right\} \tag{29}$$

The total field then becomes

$$H_z = \sqrt{\frac{2}{\pi}}e^{i\pi/4}\left[e^{-ikR}\int_{-\epsilon_1}^{\infty}\frac{e^{-i\lambda^2}}{\sqrt{(\lambda^2+2kR)}}\,d\lambda + e^{-ikS}\int_{-\epsilon_2}^{\infty}\frac{e^{-i\lambda^2}}{\sqrt{(\lambda^2+2kS)}}\,d\lambda\right]. \tag{30}$$

5. The nature of the solution

If in the first and second terms of (30) we make the respective substitutions

$$\lambda = \sqrt{(2kR)}\mathrm{sh}(\tfrac{1}{2}\xi), \qquad \lambda = \sqrt{(2kR)}\mathrm{ch}(\tfrac{1}{2}\xi)$$

the solution (30) goes over to Macdonald's form (4). For the purpose of computation, however, (30) is the most satisfactory form. It is completely analogous to that well known for an incident plane wave; and to see how (30) goes over to this latter solution when the line-source is taken to infinity we need only multiply by $\sqrt{(kr_0)}\exp(ikr_0)$ and let $r_0\to\infty$. This gives the famous result

$$\frac{e^{i\pi/4}}{\sqrt{\pi}}\{e^{ikr\cos(\theta-\theta_0)}F[-\sqrt{(2kr)}\cos\tfrac{1}{2}(\theta_0-\theta)]+e^{ikr\cos(\theta+\theta_0)}F[-\sqrt{(2kr)}\cos\tfrac{1}{2}(\theta_0+\theta)]\}, \tag{31}$$

where

$$F(a) = \int_a^{\infty}e^{-i\lambda^2}\,d\lambda. \tag{32}$$

The diffraction term (27) may be written

$$\mp\sqrt{\frac{2}{\pi}}e^{i\pi/4}e^{-ikR}\int_{\sqrt{(1-R/R_1)}}^{\infty}\frac{e^{-ikR_1\lambda^2}}{\sqrt{(\lambda^2+2R/R_1)}}\,d\lambda; \tag{33}$$

and providing only that kR_1 is large compared to unity, an asymptotic approximation to (33) is obtained by putting λ equal to its lower limit value in the non-exponential factor of the integrand (7). This gives

$$\mp\sqrt{\frac{2}{\pi}}e^{i\pi/4}\frac{e^{-ikR}}{\sqrt{\{k(R_1+R)\}}}F[\sqrt{\{k(R_1-R)\}}]. \tag{34}$$

The approximation to (26) under the same conditions is obtained on replacing R by S in (34). These approximations give the complete field in terms of the tabulated Fresnel integral, as for the case of an incident plane wave, and they are only invalid if both the source and the point of observation are well within a wave-length of the diffracting edge.

If $k(R_1-R)$ is large, the asymptotic expansion

$$e^{ia^2}F(a)\sim\frac{1}{2ia}$$

enables (34) to be written in the form

$$-\frac{e^{-i\pi/4}}{2\sqrt{(2\pi)}\cos\frac{1}{2}(\theta_0-\theta)}\frac{e^{-ikr_0}}{\sqrt{(kr_0)}}\frac{e^{-ikr}}{\sqrt{(kr)}}. \tag{35}$$

This and the analogous expression with $-\theta_0$ replacing θ_0 represent the diffraction field as a cylindrical wave diverging from the diffracting edge. It is interesting to note that the representation (35) is valid everywhere except in the region close to the shadow edge $\theta-\theta_0=\pi$ bounded by a hyperbola

$$R-r=\text{constant}.$$

This hyperbola is the counterpart of the parabola well known in the case of an incident plane wave (see e.g. **6**, p. 145).

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CONTENTS

R. V. SOUTHWELL: On the Analogues relating Flexure and Extension of Flat Plates 257

H. R. HASSÉ: The Bending of a Uniformly Loaded Clamped Plate in the form of a Circular Sector 271

M. ROTHMAN: Isolated Force Problems in Two-dimensional Elasticity (I) 279

S. GOLDSTEIN and M. J. LIGHTHILL: A Note on the Hodograph Transformation for the Two-dimensional Vortex Flow of an Incompressible Fluid 297

M. J. LIGHTHILL: Reflection at a Laminar Boundary Layer of a Weak Steady Disturbance to a Supersonic Stream, neglecting Viscosity and Heat Conduction 303

F. URSELL and G. N. WARD: On Some General Theorems in the Linearized Theory of Compressible Flow 326

L. C. WOODS: Improvements to the Accuracy of Arithmetical Solutions to certain two-dimensional Field Problems . . 349

F. G. FRIEDLANDER: On the Forced Vibrations of Quasi-linear Systems 364

P. C. CLEMMOW: A Note on the Diffraction of a Cylindrical Wave by a Perfectly Conducting Half-plane 377

ICS
1950
257
271
279
297
303
326
349
364
377